

Contributions to Potential Scattering

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English translation and scientific updating with GPT.

Spanish original: [./tesis.pdf](#)

To my parents, Alejandro and María Pilar;
and to my little sisters, Pilar and Yolanda.

About this edition

This edition carries the five chapters of the thesis into English, with their examples, calculations and conversational habits intact. The Spanish preface dates the research to May 1995; the surviving Spanish PDF was typeset again in June 2001. The link on the title page points to that original file.

The equations have been recalculated, conventions made consistent, and the bibliography brought up to date. The mark **(2026)** identifies literature connections and established developments; **(2026-gpt)** identifies additional calculations, constructions and improvements developed for this edition. These marks identify editorial provenance; historical priority is assessed separately. The accompanying references identify the mathematical background. Ordinary corrections of signs, factors, units and names are incorporated directly into the text.

The main convention throughout is $\hbar^2/(2m) = 1$. Chapters 1–3 and 5 use transmission amplitudes on the diagonal of the scattering matrix. Chapter 4 also uses a folded boundary-port ordering, with its dictionary stated explicitly. All numerical figures in this edition are generated from the equations stated in the text. Their computation is described in the appendices and supplied with the source. The restored photographic diagrams and the [monochrome figure archive](#) preserve the original drawings. A concordance in Appendix A relates their original numbers to this edition. Original equation and section numbers remain reserved; an A signals additions. Appendix B accounts for every original equation and reproduces withdrawn displays with a strike-through. Chapter 5 follows the original thesis; the expanded 2026 renormalisation analysis is the separate article. Appendix C accounts for all 79 original bibliography entries and the sources identified in the chapter review.

Acknowledgements

Writing a thesis may be a personal business, but five years of work are unlikely to reach a comfortable harbour without some human support. This little note is my attempt to thank the people who supplied it.

First of all, L. J., who as supervisor had to put up with my mathematical and computing whims while keeping the stream of work somewhere near its channel.

More generally, the theoretical physics group at the University of Zaragoza, otherwise known as DFTUZ, provided a place where one could work happily and talk happily as well. Some people got considerably more of my pestering than others. Let me single out Esteve and Asorey, who endured improbable questions; Pepín and Casahorrán, who regularly asked how things were going; Alonso, whose concerns form part of one's education; the lattice people, especially Alfonso, from whom I borrowed computing time; and a few distant departmental relatives, including Mariano and Pepe Gracia, who occasionally had to endure my peculiar email style. Our third departmental arm, the efficient secretariat, removed a good many worries too.

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A very particular mention goes to the scholarship holders and precariously employed colleagues who share my generation and its concerns, both inside the department and out in the world earning a living: sing, wandering scholar, from the Pacific to the Highlands by way of the Alps.

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Preface

This thesis studies several problems in modern potential scattering, gathering work carried out during my doctoral years. Its backbone is the S matrix and its symmetries. Already for motion on the full real line, that little matrix begins to advertise the structural richness it acquires in theories with more freedom. We pay particular attention to changes of scale and to supersymmetry.

The introduction supplies the details. Here let me say, with some optimism, that the work has been enormously educational. Over five years, contributions from the supervisor, the department, courses, conferences, trains and fevers became mixed in a succession which I suppose deserves to be called formative. It certainly was in this case.

The actual investigation used numerical and analytical techniques. Among the latter, connections bounced from algebra to geometry according to need or point of view. Much of what I learned is reflected in these pages. I hope to pass it on with a tolerable amount of confusion, and with roughly the same amazement that accompanied the learning. The most astonishing thing about a thesis, however, is the mere flow of its production.

Ours began with an alternation between soldering irons, Alfonso's transputers, soliton drawings, Mathematica and KdV equations. We then settled on scattering on the line, calculating bound states with Fortran integrators. In 1992 Rodolfo Wehrhahn showed us his scattering theory and we jumped towards calculations with fewer numerical ingredients. There was a pause, some courses away, and a leisurely procession of poles and deltas, with Mathematica returning to the scene.

Then things began to fall into place like ripe fruit. In December 1993, after some Compostela lessons on renormalisation, the boss dropped a paper by Rajeev and Gupta on my desk. Soon afterwards came a short note by Zhao and an argument about delta-prime interactions. Eduardo, by then sharing the office, helped take that argument apart with the appropriate rigour. Finally, in Paris in August 1994, I ran into Pavel and his newly completed thesis. Its subject had fortunately diverged from this one. His rigorous distribution theory had reached a stage that we had only begun to sketch in our office.

In the intervals, a couple of seminars through which I managed to stay awake, and some cold summer courses, occasionally amounting to a Silesian winter, allowed further ideas to slip into the work. Then came the business of writing, arranging the ideas and watching the $\text{\TeX}$ grow until there was a thesis in one's hands. Looking through the pages, one gets the extraordinary feeling of having come within a few millimetres of thematic unity.

Once that feeling subsides, the tension between unifying and diversifying becomes manageable. One converges weakly towards a point at which covers can be added and the final word written. Credit Nature and the direction of the work for its unifying features; the remaining disorder belongs to me. I hope that at least a small section here and there repays the reader's company.

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Introduction

*For eighty pages or so,
forty cigarettes will do.*

Bertolt Brecht, *Refugee Conversations*

.1 General concepts of scattering

Scattering theory compares states entering and leaving an interaction region. It lets us learn about an object through indirect measurements, which explains its place among the most widely used tools of dynamics. The general idea works in classical mechanics, quantum mechanics and quantum field theory. Collision experiments give the last of these a particularly direct reason to take it seriously.

The basic ingredients are a family of free states and a dynamics with interaction. We compare the two where the interaction has become negligible, usually at large distances or at remote times. Two comparisons make one scattering experiment: a free incoming state is associated with an interacting state, and that interacting state is associated with a free outgoing state. Compose the associations and the detailed history inside the interaction region has disappeared from the description. What remains is the relation between free past and free future.

In quantum mechanics the states live in a Hilbert space, often $L^2(\mathbb{R})$, and the scattering operator is unitary on the appropriate free subspace. Its energy fibres form the scattering matrix. This operator carries much of the load in the thesis. We build the wave functions explicitly when they are useful, and let the matrix do the bookkeeping when they have become baggage.

The familiar machinery of three-dimensional scattering—Jost functions, phase shifts, partial waves and inverse equations—has close relatives on the line. Using those relatives gives us a common language for factorisation, geometry, poles and singular limits. The parity decomposition of a symmetric full-line potential makes the relation with radial scattering especially transparent.

.2 Scattering in one dimension

Many features of scattering are already present in one dimension. A rotationally invariant three-dimensional problem reduces, after angular decomposition, to half-line problems. On the full line we retain an intermediate amount of freedom: the particle may arrive from either side, and it may leave on either side. The scattering matrix has four entries. This is enough room for a surprisingly lively collection of symmetries and boundary conditions.

We use the Deift–Newton formulation to organise the direct and inverse problems. Jost

solutions encode the behaviour at infinity, regular solutions provide a convenient reference at a finite point, and physical solutions display the incoming, reflected and transmitted waves. The Jost matrix connects these descriptions. Parity makes it diagonal and recovers the familiar half-line Jost functions.

Chapter 1 collects this machinery and a catalogue of examples. The ordinary delta and a pair of deltas make quick tests of the normalisations. The exponential well takes us into special functions. Compact support and weighted decay lead to different continuation and threshold properties, which later become important when a family of potentials is squeezed into a point.

.3 SUSY quantum mechanics: scattering and bound states

Supersymmetric quantum mechanics gives us a useful way to add or remove a lowest bound state while controlling the rest of the spectrum. A first-order operator intertwines a pair of Hamiltonians, and its limits at infinity tell us how the scattering amplitudes change. Shape invariance turns a succession of such pairs into a ladder.

In Chapter 2 the ladder grows out of geometry. The radial Laplacian of a rank-one symmetric space contains a first derivative fixed by the radial volume density. Removing that derivative produces a superpotential. The factorisation is therefore already sitting in the measure, waiting for us to write it down. We compare compact, noncompact and Euclidean spaces in this language and calculate their isotropic spectra.

One small warning belongs to the working method itself: an operator includes its domain. The same differential expression on a line, a half-line or a finite interval describes different quantum systems. We keep the endpoint conditions attached to the formulas as we climb the ladders.

.4 Changes of scale and their effects

The algebraic method is very good at hiding the choice of units under the carpet. To see the scales again we vary a coupling continuously. Bound states move, thresholds are crossed, resonances approach one another, and an apparently gentle parameter change can alter the analytic structure of the S matrix.

.5 Trajectories and poleology

Chapter 3 follows those movements. We begin with the discrete energies of several regular and singular wells and compare their strong-coupling behaviour. Then we leave the positive imaginary momentum axis and inspect the resonance poles themselves. One delta is an economical starting point: its single pole moves straight up the imaginary axis. Two and three deltas already produce coalescences, changes of branch and long excursions through the lower half-plane. A square well and a reflectionless potential add useful comparisons.

(2026) The local behaviour of a regular minimum determines its subleading strong-coupling scaling. A smooth quadratic bottom, a cusp and a flat bottom give different corrections. Double resonance zeros have a square-root unfolding and fit naturally into the theory of exceptional points of the outgoing resonance problem. These connections enter directly where the corresponding calculations occur.

.6 Contact interactions

Chapter 4 concentrates on operators that agree with the free Hamiltonian everywhere away from the origin. Their complete classification is obtained from the self-adjoint extensions of the free operator on $\mathbb{R} \setminus \{0\}$. Conservation of probability gives the same boundary relations. Both methods lead to four real parameters.

The ordinary delta occupies one line in this family. Two other simple lines have often shared the name delta prime. We identify them by their boundary conditions. The derivative-continuous contact satisfies

$$\psi'(0+) = \psi'(0-), \quad \psi(0+) - \psi(0-) = B\psi'(0-), \quad (\text{I.A1})$$

and is denoted by δ'_{int} ; the old symbol $\delta^{(0)}$ is mentioned when connecting to the Spanish text. The scale-invariant contact instead has

$$\psi(0+) = \theta\psi(0-), \quad \psi'(0+) = \theta^{-1}\psi'(0-). \quad (\text{I.A2})$$

The constants B and θ have different dimensions, and their renormalisation trajectories immediately separate their physical roles.

We examine regularising potentials, distributions acting on discontinuous test functions, first-order factorisations and operator-based distances. Each description contributes a different piece of the picture. The relation between a boundary condition and a microscopic family is treated through operator convergence, with the zero-energy resonance kept explicitly in the calculation.

.7 Renormalisation and point-supported potentials

Chapter 5 lets the scale act directly on the scattering matrix:

$$(T_t S)(k) = S(e^{-t}k). \quad (\text{I.A3})$$

The scale-independent contacts are fixed points. Marginal directions move along a fixed surface, while relevant and irrelevant directions connect its points to the separated Neumann and Dirichlet contacts. The construction was publicly presented in 1994 [1].

.8 Further connections

(2026) The chapter updates connect inverse scattering with discrete norming data, the geometric ladders with hypergeometric functions, resonance coalescences with their local square-root form, and point contacts with modern microscopic approximations. The original final chapter keeps its scattering-space construction and regularisation examples together. The separate 2026 article develops their broader renormalisation and action-theoretic connections.

Chapter 1

Scattering in one dimension

Wave operators and stationary scattering. Jost and regular solutions. The two-channel matrix. Parity and time reversal. Compact support, thresholds and tails. Inverse scattering, norming constants and the little miracle of consistency.

1.1 General orientation

Our free Hamiltonian is

$$H_0 = -\frac{d^2}{dx^2}, \quad \mathcal{D}(H_0) = H^2(\mathbb{R}). \quad (1.1)$$

For the first part of the chapter take $H = H_0 + V$, with V real and sufficiently decaying. A convenient working hypothesis is

$$V \in L^1_1(\mathbb{R}), \quad \int_{\mathbb{R}} (1 + |x|)|V(x)| dx < \infty. \quad (1.A1)$$

More moments give more differentiability at threshold; stronger tails give a larger domain of analytic continuation. We state the extra assumption where it is used. Point interactions have their own operator domains and arrive in Chapter 4.

Write the incoming and outgoing wave operators as

$$\Omega_{\text{in}} = s - \lim_{t \rightarrow -\infty} e^{itH} e^{-itH_0}, \quad \Omega_{\text{out}} = s - \lim_{t \rightarrow +\infty} e^{itH} e^{-itH_0}. \quad (1.2)$$

Under the usual short-range assumptions they are complete: their ranges equal the absolutely continuous subspace of H . The scattering operator

$$S = \Omega_{\text{out}}^\dagger \Omega_{\text{in}} \quad (1.3)$$

is unitary on the free Hilbert space and commutes with H_0 . At each positive energy $E = k^2$ there are two propagation directions. In our transmission-diagonal ordering,

$$S(k) = \begin{pmatrix} t_L(k) & r_R(k) \\ r_L(k) & t_R(k) \end{pmatrix}. \quad (1.4)$$

The labels refer to the side from which the particle arrives.

For incidence from the left, the stationary solution is normalised by

$$\psi_L(x, k) \sim \begin{cases} e^{ikx} + r_L(k)e^{-ikx}, & x \rightarrow -\infty, \\ t_L(k)e^{ikx}, & x \rightarrow +\infty. \end{cases} \quad (1.5)$$

For incidence from the right,

$$\psi_R(x, k) \sim \begin{cases} t_R(k)e^{-ikx}, & x \rightarrow -\infty, \\ e^{-ikx} + r_R(k)e^{ikx}, & x \rightarrow +\infty. \end{cases} \quad (1.7)$$

The forward-scattering quantities of the original notation are

$$\sigma_r = |r|^2, \quad \sigma_{\text{forward}} = |t - 1|^2. \quad (1.6)$$

The measurable reflection and transmission probabilities are $|r|^2$ and $|t|^2$. The quantity $|t - 1|^2$ is also useful when describing the forward scattered wave relative to free propagation. Keeping those two roles separate saves a surprisingly common misunderstanding.

The stationary equation is

$$-u''(x) + V(x)u(x) = k^2u(x). \quad (1.8)$$

The current in our units is $j = 2 \operatorname{Im}(\bar{u}u')$. Its conservation gives the unitary relations between the entries of [equation \(1.4\)](#). This direct stationary route and the wave-operator route describe the same experiment once the asymptotic normalisation is fixed.

1.2 The Deift–Newton formulation

There are several efficient ways of packaging scattering on the line. Faddeev’s construction and the Deift–Trubowitz analysis provide the basic inverse theory [2, 3]. Newton’s matrix formulation keeps the resemblance to radial three-dimensional scattering especially visible [4, 5, 97]. That resemblance is useful for our purposes: we can carry familiar Jost-function manipulations into a two-channel setting with very little extra luggage.

1.2.1 Special solutions of the Schrödinger equation

Jost solutions. Define f_+ and f_- by their behaviour at the two ends:

$$f_+(x, k) \sim e^{ikx} \quad (x \rightarrow +\infty), \quad f_-(x, k) \sim e^{-ikx} \quad (x \rightarrow -\infty). \quad (1.9)$$

The corresponding Volterra equations are

$$f_+(x, k) = e^{ikx} + \int_x^\infty \frac{\sin k(y-x)}{k} V(y) f_+(y, k) dy, \quad (1.11)$$

$$f_-(x, k) = e^{-ikx} + \int_{-\infty}^x \frac{\sin k(x-y)}{k} V(y) f_-(y, k) dy. \quad (1.12)$$

At $k = 0$ the sine quotient is replaced by its continuous limit. Each equation builds the solution from the end at which its normalisation is specified. Successive substitution is often enough to establish the analyticity or asymptotics that one needs.

Physical solutions. The outgoing free resolvent has kernel

$$(H_0 - k^2)^{-1}(x, y) = \frac{i}{2k} e^{ik|x-y|}, \quad \operatorname{Im} k > 0. \quad (1.14)$$

Thus the left-incident Lippmann–Schwinger equation is

$$\psi_L(x, k) = e^{ikx} - \frac{i}{2k} \int_{\mathbb{R}} e^{ik|x-y|} V(y) \psi_L(y, k) dy, \quad (1.13)$$

with e^{-ikx} as the incident term for ψ_R . The two physical solutions are related to the Jost solutions by

$$\psi_L = t_L f_+, \quad \psi_R = t_R f_-. \quad (1.A2)$$

The Jost solution has deliberately forgotten the transmitted amplitude; the physical solution puts it back.

Regular solutions. At an ordinary point, take u_1, u_2 with

$$u_1(0, k) = u_2(0, k) = 1, \quad u_1'(0, k) = ik, \quad u_2'(0, k) = -ik. \quad (1.15)$$

For example,

$$u_1(x, k) = e^{ikx} + \int_0^x \frac{\sin k(x-y)}{k} V(y) u_1(y, k) dy. \quad (1.16)$$

These initial data make the regular solution a convenient pivot. Its reference point stays in place while the asymptotic solutions are compared from opposite ends. For $k \neq 0$ the pair is a basis of solutions.

Use row vectors $u = (u_1, u_2)$ and $\psi = (\psi_L, \psi_R)$, and define the Jost matrix by

$$u(x, k) = \psi(x, k) J(k). \quad (1.17)$$

Any other fixed regular basis gives a corresponding change of Jost coordinates. Individual matrix entries are basis dependent; their role in the scattering relation is invariant under a consistent change of coordinates.

Let

$$q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (1.A3)$$

The regular initial data imply $u(x, k) = u(x, -k)q$, while the physical asymptotics imply $\psi(x, k) = \psi(x, -k)qS(k)$. Substituting into equation (1.17) gives

$$\boxed{S(k) = qJ(-k)qJ(k)^{-1}}. \quad (1.25)$$

The familiar quotient of Jost functions has become a matrix quotient. The order of factors is part of the formula.

For real scalar potentials an equivalent scalar shortcut uses the Wronskian $\mathcal{W}(u, v) = uv' - u'v$:

$$t(k) = \frac{2ik}{\mathcal{W}(f_-, f_+)}. \quad (1.A4)$$

At the free potential the Wronskian is $2ik$, so the normalisation checks itself. The reflection coefficients are the other connection coefficients in the same two solution bases.

1.2.2 Elementary properties of the matrix

Unitarity. For real $k \neq 0$,

$$S^\dagger S = SS^\dagger = \mathbb{I}. \quad (1.26)$$

In particular,

$$|t_L|^2 + |r_L|^2 = 1, \quad |t_R|^2 + |r_R|^2 = 1, \quad \overline{t_L} r_R + \overline{r_L} t_R = 0. \quad (1.A5)$$

The last relation contains phase information that probabilities alone discard.

Reflection in momentum. For a real scalar potential, conjugating the equation and its normalisations gives

$$S(-k) = \overline{S(k)} \quad (k \in \mathbb{R}). \quad (1.27)$$

Analytic continuation then pairs resonance poles at k and $-\bar{k}$ wherever that continuation exists. This is the symmetry about the imaginary axis that appears in the pole plots of Chapter 3.

For a general energy-independent self-adjoint boundary relation, the momentum-reversal identity in the transmission-diagonal convention is

$$S(-k)^\dagger = qS(k)q. \quad (1.28)$$

Its port-ordered version becomes the particularly simple identity $\mathcal{S}(-k) = \mathcal{S}(k)^\dagger$.

Reciprocity. Canonical time reversal exchanges incoming and outgoing waves and reverses momentum. In this ordering it implies

$$S(k)^T = qS(k)q, \quad (1.31)$$

or $t_L = t_R$. A real multiplication potential has this symmetry. A localised gauge phase is treated explicitly in the point-interaction chapters. The word reciprocity here refers to time reversal and the associated channel relation.

1.2.3 High and low energies

For an integrable scalar potential the high-energy amplitudes approach

$$t(k) \rightarrow 1, \quad r_L(k), r_R(k) \rightarrow 0. \quad (1.A6)$$

The leading Born terms, when the relevant integrals are controlled, are

$$t(k) = 1 - \frac{i}{2k} \int V(x) dx + o(k^{-1}), \quad (1.A7)$$

$$r_L(k) = -\frac{i}{2k} \int e^{2ikx} V(x) dx + o(k^{-1}). \quad (1.A8)$$

This asymptotic freedom has a concrete range of applicability: the potential is held fixed while momentum grows. A zero-range limit changes the potential at the same time. Uniformity in that second parameter becomes a separate question.

At low energy, [equation \(1.A1\)](#) gives a useful generic/exceptional alternative. Generically, the zero-energy equation has no nonzero bounded solution on the full line, and

$$t(k) = O(k), \quad r_L(k), r_R(k) \rightarrow -1, \quad S(0) = -q. \quad (1.33)$$

At an exceptional coupling there is a bounded zero-energy solution, and transmission can approach a nonzero value. In the usual short-range class that bounded solution is a threshold resonance: it approaches constants at infinity and lies outside $L^2(\mathbb{R})$. Its ratio of limiting constants becomes a coordinate of the scale-invariant contact in Chapter 4.

The distinction is visible directly in [equation \(1.A4\)](#). A nonzero limiting Wronskian gives linear transmission; a vanishing one permits the exceptional limit. In customary normalisations the generic expression is

$$t(k) = -2ik/\gamma + o(k), \quad (1.34)$$

with γ determined by the zero-energy solution.

Power-law tails can change the threshold exponents. For example, tails of the form

$$V(x) = \frac{\rho^2 - 1/4}{(1+x)^2} \mathbf{1}_{x>0} + \frac{\sigma^2 - 1/4}{(1-x)^2} \mathbf{1}_{x<0} \quad (1.35)$$

lead, in nondegenerate parameter regimes, to transmission with a power governed by $\rho + \sigma$ [6, 7, 129]. The high-energy limit can remain familiar while the threshold behaviour has acquired quite a different character. Weighted decay classes are therefore more informative than a single informal label such as short range.

(2026) Levinson's theorem provides a global accounting of these endpoint effects. Its topological formulation for point interactions expresses the number of bound states as a winding number, with the threshold and high-energy corrections supplied by the remaining boundary pieces of the wave-operator construction [8]. The scattering phase along positive energy is one part of that complete contour.

1.2.4 Functional and asymptotic descriptions

The scattering operator can be represented after choosing spectral coordinates for the comparison dynamics. The stationary amplitudes in equations (1.5) and (1.7) fix those coordinates through their plane-wave normalisation. Changing the phases of spectral basis vectors changes the displayed matrix by the corresponding incoming and outgoing unitary transformations.

Newton's discussion of functional and asymptotic matrices is best read with that normalisation visible [5]. A factor such as $g(k)^*/g(k)$ can appear when the two descriptions use different phase choices. Fixing the physical comparison states fixes which amplitudes describe the experiment. This is the same familiar freedom encountered when changing the phase of an eigenvector, now applied separately across a continuous spectrum.

1.3 A catalogue of useful cases

1.3.1 Time reversal

With $t_L = t_R = t$, the two physical solutions differ from f_+, f_- by the same transmission factor. The Jost matrix can then be calculated from Wronskians between the regular and Jost bases. A connection coefficient is simply the appropriate ratio of Wronskians: if $v = Au + Bw$, then $A = \mathcal{W}(v, w)/\mathcal{W}(u, w)$.

This little identity is often a quicker tool than solving a large matching system. It also makes analytic continuation transparent, because the Wronskians are independent of position and can be evaluated where their formulas are simplest.

1.3.2 Parity

If $V(x) = V(-x)$, then $f_-(x, k) = f_+(-x, k)$ and

$$S(k) = \begin{pmatrix} t & r \\ r & t \end{pmatrix}. \quad (1.42)$$

The constant parity change of basis diagonalises it:

$$S_e = t + r = e^{2i\delta_e}, \quad S_o = t - r = e^{2i\delta_o}. \quad (1.40)$$

The labels e, o are useful here: supersymmetric partners and one-sided limits also need plus and minus signs later on.

On $x > 0$ the even regular solution has $u_e(0) = 1, u'_e(0) = 0$, while the odd one can be taken with $u_o(0) = 0, u'_o(0) = 1$. The odd half-line problem is exactly the three-dimensional reduced s wave with its usual regular endpoint condition. The even problem supplies the extra channel which the full line retains.

1.3.3 Parity and time reversal together

For an ordinary nonsingular potential at the origin, the parity Jost functions can be normalised as

$$J_o(k) = f_+(0, k), \quad (1.50)$$

$$J_e(k) = \frac{f'_+(0, k)}{ik}. \quad (1.51)$$

Both equal one for free motion. Their quotients give

$$S_e(k) = \frac{J_e(-k)}{J_e(k)}, \quad S_o(k) = \frac{J_o(-k)}{J_o(k)}. \quad (1.49)$$

A point interaction at the endpoint changes the regular boundary condition and hence the corresponding Jost functional. This is especially easy to see for the delta.

One delta. Let $V = g\delta(x)$. The wave function is continuous and its derivative jumps by $g\psi(0)$. On the even half-line the condition is $\psi'(0+) = g\psi(0)/2$. With $f_+(x, k) = e^{ikx}$ for $x > 0$,

$$J_e(k) = \frac{f'_+(0, k) - (g/2)f_+(0, k)}{ik} = 1 + \frac{ig}{2k}, \quad (1.53)$$

$$J_o(k) = 1. \quad (1.54)$$

Consequently

$$t(k) = \frac{2ik}{2ik - g}, \quad r(k) = \frac{g}{2ik - g}. \quad (1.49)$$

The pole is at $k = -ig/2$. Attractive $g < 0$ gives the bound state $E = -g^2/4$; the odd wave passes through the origin with zero amplitude there and supplies its free channel.

Two deltas. For

$$V(x) = g\{\delta(x - a) + \delta(x + a)\}, \quad a > 0, \quad (1.55)$$

the right Jost solution follows immediately from [equation \(1.11\)](#):

$$f_+(x, k) = e^{ikx} - \frac{g}{k} \sin k(x - a)f_+(a, k)\mathbf{1}_{x < a} - \frac{g}{k} \sin k(x + a)f_+(-a, k)\mathbf{1}_{x < -a}. \quad (1.56)$$

Evaluate between the two centres and then at zero:

$$J_o(k) = 1 + \frac{g}{k} e^{ika} \sin ka, \quad (1.57)$$

$$J_e(k) = 1 + \frac{ig}{k} e^{ika} \cos ka. \quad (1.58)$$

We have solved the direct problem almost before assembling a matching matrix. These functions reappear in the pole chapter, where they earn their keep a second time.

The exponential well. Take $V(x) = -ge^{-|x|}$ with $g > 0$. On the right, put $z = 2\sqrt{g}e^{-x/2}$. The equation becomes Bessel's equation with order $-2ik$. The normalised outgoing solution is

$$f_+(x, k) = g^{ik}\Gamma(1 - 2ik)J_{-2ik}(2\sqrt{g}e^{-x/2}), \quad x \geq 0. \quad (1.64)$$

The small- z expansion gives e^{ikx} , which is a useful check of the order and the prefactor. The parity functions are

$$J_o(k) = g^{ik}\Gamma(1 - 2ik)J_{-2ik}(2\sqrt{g}), \quad (1.65)$$

$$J_e(k) = -\frac{\sqrt{g}}{ik}g^{ik}\Gamma(1 - 2ik)J'_{-2ik}(2\sqrt{g}). \quad (1.66)$$

At $k = i\kappa$ the orders become 2κ . Odd and even bound levels are therefore zeros, respectively, of $J_{2\kappa}(2\sqrt{g})$ and its derivative with respect to its argument. The solution also displays why special-function singularities and physical bound-state poles need to be examined together with their possible cancellations [9, 10, 11].

1.3.4 Compact support and logarithmic derivatives

Suppose V vanishes beyond $|x| = b$. Propagate the regular solutions to b and set

$$h_e(k) = \frac{u'_e(b, k)}{u_e(b, k)}, \quad h_o(k) = \frac{u'_o(b, k)}{u_o(b, k)}. \quad (1.A10)$$

Outside the support write $u_e \propto \cos(kx + \delta_e)$ and $u_o \propto \sin(kx + \delta_o)$. Matching their logarithmic derivatives gives

$$S_e(k) = e^{-2ikb} \frac{k - ih_e(k)}{k + ih_e(k)}, \quad (1.68)$$

$$S_o(k) = e^{-2ikb} \frac{h_o(k) + ik}{h_o(k) - ik}. \quad (1.70)$$

Poles of a logarithmic derivative at a zero of u are coordinate singularities; the corresponding ratio remains well defined by its un-divided expression. For the square well we keep those expressions explicitly, which avoids introducing unnecessary cancelling zeros.

The location of the interaction has an equally simple effect. Translating $V(x)$ to $V(x - a)$ leaves t unchanged and gives

$$r_L(k) \mapsto e^{2ika}r_L(k), \quad r_R(k) \mapsto e^{-2ika}r_R(k). \quad (1.72)$$

Reflection remembers where the target was. A reflectionless potential has already discarded that positional information from its real-axis scattering amplitudes.

1.3.5 Short range, effective range and medium-range tails

For even potentials with sufficient decay and away from a singular threshold parametrisation, introduce the one-dimensional effective-range coefficients through

$$k \cot \delta_o(k) = -\frac{1}{a_o} + \frac{1}{2}r_o k^2 + O(k^4), \quad (1.73)$$

$$k \tan \delta_e(k) = \frac{1}{a_e} + \frac{1}{2}r_e k^2 + O(k^4). \quad (1.74)$$

The even scattering length here is the zero of the extrapolated zero-energy even wave, $u_e \propto 1 - x/a_e$. In particular the delta has $a_e = -2/g$. Stating this convention explicitly is more economical than arguing later about a minus sign in a scattering length.

For compact support the Jost solutions have continuation properties inherited from finite-interval Volterra equations. For exponentially decaying tails the natural continuation initially reaches a strip below the real axis. Further meromorphic continuation can be available from an exact formula, as in the exponential well. The size of a decay exponent determines a continuation region; the word exponential by itself supplies no entire-function theorem for the scattering matrix.

More generally, define

$$L^1_\sigma(\mathbb{R}) = \left\{ V : \int_{\mathbb{R}} (1 + |x|^\sigma) |V(x)| dx < \infty \right\}. \quad (1.75)$$

As moments are lost, derivatives at $k = 0$ and simple effective-range expansions can be lost with them. Newton and Aktosun give useful examples and detailed threshold classifications [6, 12, 7, 13]. The inverse-square tail sits at a particularly instructive boundary: it scales like the kinetic term and brings its own threshold exponents.

(2026) Effective-range coefficients also carry causality information. Wigner’s time-delay bound and its later extensions constrain which energy dependence is compatible with a specified interaction range [14, 15]. Those constraints become especially useful when a formal scattering matrix is proposed as the limit of a microscopic potential.

1.4 Inverse scattering

The inverse equations stay in the background of several later examples, so let us put the main tools on the table. We use a form with the data specified explicitly. A full-line reconstruction takes a reflection coefficient, the bound-state energies and suitable positive norming constants. The remaining scattering amplitudes are then connected by unitarity and analyticity under the hypotheses of the inverse theorem [3, 4, 16].

1.4.1 Reduction by bound states

A Darboux transformation adds or removes a discrete Jost zero through a Blaschke-type factor. On the full line the reflection and transmission amplitudes also acquire the corresponding asymptotic factors; Chapter 2 derives them directly from the first-order intertwiner. For a reflectionless system, addition of a state at $i\kappa$ multiplies the transmission amplitude by

$$\frac{k + i\kappa}{k - i\kappa}. \quad (1.77)$$

The norming constant controls the position or relative placement of the associated bound-state structure. Its presence is already a reminder that pole locations alone carry only part of the reconstruction data.

One can solve the inverse problem after removing the bound states and restore them afterwards, or incorporate the discrete terms into the inverse kernel from the beginning. The latter is particularly compact in the Marchenko equation.

1.4.2 Gel'fand–Levitan reconstruction

For a half-line regular problem let $u_0(x, \lambda)$ denote the free regular solution and ρ, ρ_0 the interacting and free spectral measures, using the same endpoint normalisation. Form

$$G(x, y) = \int u_0(x, \lambda) u_0(y, \lambda) d(\rho - \rho_0)(\lambda). \quad (1.83)$$

With the transformation convention

$$u(x, \lambda) = u_0(x, \lambda) + \int_0^x K(x, s) u_0(s, \lambda) ds, \quad (1.A11)$$

the Gel'fand–Levitan equation is

$$K(x, y) + G(x, y) + \int_0^x K(x, s) G(s, y) ds = 0, \quad 0 \leq y \leq x, \quad (1.81)$$

and

$$V(x) = 2 \frac{d}{dx} K(x, x). \quad (1.84)$$

The discrete spectral masses are part of ρ , so they enter along with the continuum density. In the full-line matrix formulation the two regular initial conditions give a matrix spectral measure and the corresponding matrix transformation kernel [4, 5]. The finite interval of integration is one practical attraction of this route.

1.4.3 Marchenko reconstruction

For the right reconstruction, let $r_R(k)$ be the right reflection coefficient and let $\kappa_n > 0, c_n > 0$ be the discrete parameters in the corresponding right Jost normalisation. Set

$$F(s) = \frac{1}{2\pi} \int_{\mathbb{R}} r_R(k) e^{iks} dk + \sum_{n=1}^N c_n^2 e^{-\kappa_n s}. \quad (1.87)$$

The kernel $K(x, y)$ solves

$$K(x, y) + F(x + y) + \int_x^\infty K(x, z) F(z + y) dz = 0, \quad y \geq x, \quad (1.88)$$

and the reconstructed potential is

$$V(x) = -2 \frac{d}{dx} K(x, x). \quad (1.89)$$

The change of sign from the previous reconstruction follows from using a transformation based at infinity. Once the convention for the transformation kernel is fixed, the sign is fixed with it.

The miracle. Newton's matrix presentation reconstructs the line through two coupled components. A compatible scalar potential must emerge from both. The agreement looks sufficiently surprising, when the final formula is viewed alone, to have earned the name “the miracle” [4, 5]. Its content is a consistency requirement on the scattering data. Admissibility includes the analytic, symmetry and decay conditions that make the two reconstructions parts of one scalar problem.

The practical moral is pleasant enough: the inverse equations can do much of the work of deciding whether an abstract scattering matrix actually comes from the proposed class of potentials. Pointwise unitarity is one ingredient in that decision. The rest of the inverse data and the relevant regularity conditions complete it.

1.4.4 Uniqueness and the information carried by bound states

Under the standard full-line inverse hypotheses, complete scattering data determine the potential uniquely. The qualification complete matters. In a reflectionless family, real-axis reflection vanishes and

$$t(k) = \prod_{n=1}^N \frac{k + i\kappa_n}{k - i\kappa_n} \quad (1.A12)$$

contains no norming constants. Different positive norming constants therefore produce different potentials with the same displayed amplitudes.

(2026) The general finite reflectionless family has the determinant form

$$V(x) = -2 \frac{d^2}{dx^2} \log \det[\mathbb{I} + A(x)], \quad A_{mn}(x) = \frac{c_m c_n}{\kappa_m + \kappa_n} e^{-(\kappa_m + \kappa_n)x}. \quad (1.A13)$$

This is the Kay–Moses family and its inverse-scattering realisation [17, 18]. For one state, $c^2 = 2\kappa e^{2\kappa x_0}$ gives $V = -2\kappa^2 \operatorname{sech}^2 \kappa(x - x_0)$. For several states the κ_n are independent positive spectral parameters. The equally spaced poles of the Pöschl–Teller ladder form a particularly convenient subfamily.

Longer tails bring further examples of scattering ambiguity. An instructive family from the original discussion is, for $a > 0$,

$$V_a(x) = -2\delta(x) + \frac{8\mathbf{1}_{x>0}}{(1+a+2x)^2} + \frac{8\mathbf{1}_{x<0}}{(1+a^{-1}-2x)^2}. \quad (1.91)$$

Its scattering matrix is independent of a :

$$S_a(k) = \frac{1}{k+i} \begin{pmatrix} k & i \\ i & k \end{pmatrix}. \quad (1.92)$$

The inverse-square tails put this family outside [equation \(1.A1\)](#). Its SUSY partner is the repulsive delta $2\delta(x)$; the transmission is the same and both reflection amplitudes change sign [12, 13, 19].

(2026-gpt) The factorisation also displays the threshold state directly. Put

$$W_a(x) = \begin{cases} -2/(1+a+2x), & x > 0, \\ +2/(1+a^{-1}-2x), & x < 0. \end{cases} \quad (1.A14)$$

The jump of W_a is -2 , and $W_a^2 - W'_a = 2\delta(x)$ while $W_a^2 + W'_a = V_a$. A zero mode of the second operator is proportional to $\exp(\int_0^x W_a(s)ds)$. It decays as $1/|x|$ at both ends and lies in L^2 . Here the threshold state is a genuine square-integrable eigenfunction, made possible by the longer tail. The earlier bounded non-square-integrable threshold resonance belongs to the faster-decay class. The integrability assumption is doing visible work in this example.

The same phase change in reflection could be imitated at a single energy by translating a delta through a momentum-dependent distance $\pi/(2k)$. That prescription is naturally energy dependent. A fixed static translation supplies a phase proportional to k , as derived above, and keeps the physical location independent of the probing momentum.

(2026) Inverse scattering also supplies the spectral variables for nonlinear evolution. In the KdV equation, bound-state locations remain fixed while reflection and norming data acquire simple time factors [20]. The potential itself evolves nonlinearly. This is the thread connecting the early soliton calculations mentioned in the preface to the inverse machinery collected here. The later Riemann–Hilbert and nonlinear steepest-descent methods develop the same spectral organisation into precise long-time asymptotics [21].

So the first chapter leaves us with a usable set of tools. The Jost description follows poles, the parity basis separates half-line channels, the inverse equations identify the data required for reconstruction, and factorisation gives a controlled way to move between systems. The next chapter puts that last operation to work on geometry.

Chapter 2

Supersymmetry and scattering on homogeneous spaces

Thus the two went on reasoning until they reached a town where, as luck would have it, they found an algebraist.

Miguel de Cervantes, *Don Quixote*

Geometric comparison and rank-one symmetric spaces. The radial measure as a superpotential. SUSY partners and scattering amplitudes. Five elementary ladders. Compact spectra and multiplicity shifts. Hypergeometric extensions and later applications.

2.1 Geometric scattering

2.1.1 Comparing geometries

Wehrhahn proposed using the comparison principle of scattering theory to analyse Riemannian symmetric spaces [22, 23, 24]. A symmetry restricts motion. Let us take that statement quite literally: compare quantum motion with one symmetry restriction to motion with a contracted or flatter one. The difference between the two appears as a scattering operator.

Write the symmetric space as $X = G/K$. The invariant differential operators descend to the quotient, and their eigenfunctions can be compared asymptotically with the eigenfunctions of a simpler comparison algebra. One route begins with those asymptotics and reconstructs a one-dimensional potential by inverse scattering. Another begins with the invariant Laplacian and reduces it directly. The agreement between the two routes gives the construction its consistency [25, 26, 106].

The direct route remains available in compact spaces, where the endpoint problem has a discrete spectrum. In that setting the word scattering describes the common comparison method; the actual calculation is a compact spectral problem. This is convenient: the same factorisation handles the compact, noncompact and flat cases, with the geometry keeping track of the endpoint conditions.

2.1.2 Rank-one spaces and radial density

For rank one we need one radial coordinate and some angles [124, 133]. Restrict first to the isotropic sector, so the angular differential terms act trivially. Let $p = m_\alpha$ and $q = m_{2\alpha}$ be the root multiplicities. In curvature units fixed below, the noncompact radial volume density

is

$$J(r) = C(\sinh r)^p(\sinh 2r)^q. \quad (2.A1)$$

The radial Laplacian is

$$\Delta_r = \partial_r^2 + \frac{J'}{J} \partial_r = \partial_r^2 + (p \coth r + 2q \coth 2r) \partial_r. \quad (2.1)$$

It acts in $L^2((0, \infty), J(r)dr)$. This weighted Hilbert space is part of the geometry, so let us carry it along before doing any algebra.

The unitary substitution $u = J^{1/2}\psi$ gives

$$J^{1/2}(-\Delta_r)J^{-1/2} = -\partial_r^2 + W^2 + W', \quad W = \frac{1}{2} \frac{J'}{J}. \quad (2.2)$$

Thus

$$W = \frac{p}{2} \coth r + q \coth 2r. \quad (2.A2)$$

The superpotential has come directly from the measure. The first derivative has disappeared because the radial density has been absorbed into the wave function.

It is useful to put

$$A = \frac{p+q}{2}, \quad B = \frac{q}{2}, \quad \rho = A + B = \frac{p}{2} + q. \quad (2.A3)$$

Since $2 \coth 2r = \coth r + \tanh r$, we have

$$\begin{aligned} W &= A \coth r + B \tanh r, \\ W^2 + W' &= \rho^2 + A(A-1) \operatorname{csch}^2 r - B(B-1) \operatorname{sech}^2 r. \end{aligned} \quad (2.A4)$$

Equivalently, after subtracting the constant ρ^2 ,

$$V(r) = \frac{p(p+2q-2)}{4 \sinh^2 r} + \frac{q(q-2)}{\sinh^2 2r}. \quad (2.3)$$

The constant shift places the continuum threshold at zero in the Schrödinger coordinate. The original noncompact geometric operator has threshold ρ^2 .

Space	Isotropy group	p	q
$\mathbb{H}_{\mathbb{R}}^n$	$SO(n)$	$n-1$	0
$\mathbb{H}_{\mathbb{C}}^n$	$S(U(n) \times U(1))$	$2n-2$	1
$\mathbb{H}_{\mathbb{H}}^n$	$Sp(n) \times Sp(1)$	$4n-4$	3
Cayley hyperbolic plane	$Spin(9)$	8	7

Table 2.1: Root multiplicities for the noncompact rank-one spaces. Finite central quotients of the groups leave the local radial formulas unchanged.

The compact dual replaces the hyperbolic functions by circular ones:

$$J_c(r) = C(\sin r)^p(\sin 2r)^q, \quad W_c = \frac{p}{2} \cot r + q \cot 2r = A \cot r - B \tan r. \quad (2.A5)$$

On $0 < r < \pi/2$ this gives

$$H_c = -\partial_r^2 + A(A-1) \csc^2 r + B(B-1) \sec^2 r - \rho^2. \quad (2.A6)$$

For a sphere, $q = 0$ and the radial interval is $0 < r < \pi$. The expression then reduces to

$$H_{S^n} = -\partial_r^2 + \frac{(n-1)(n-3)}{4\sin^2 r} - \frac{(n-1)^2}{4}. \quad (2.6)$$

The singular-looking endpoint behaviour is the image of a smooth radial function on the manifold. It selects the geometric extension when more than one local square-integrable Schrödinger branch is available.

The Euclidean case sits between the two. Here $J(r) = r^{n-1}$ and

$$W(r) = \frac{n-1}{2r}, \quad H_{\mathbb{R}^n, \text{rad}} = -\partial_r^2 + \frac{(n-1)(n-3)}{4r^2}. \quad (2.8)$$

The comparison with a fixed half-line free convention produces a constant centrifugal phase; the familiar value is $(3-n)\pi/4$. There is no momentum scale in that phase. Comparing motion to its own Euclidean radial reference of course gives the identity comparison.

2.2 Supersymmetric quantum mechanics

The geometry has presented us with $W^2 + W'$. Let us now recall the factorisation machinery that makes this useful. A grading C and a self-adjoint supercharge Q satisfy

$$C^2 = \mathbb{I}, \quad Q^2 = \mathcal{H}, \quad \{C, Q\} = 0. \quad (2.10)$$

In the simplest one-dimensional realisation take

$$A = \partial_x + W(x), \quad A^\dagger = -\partial_x + W(x), \quad Q = \begin{pmatrix} 0 & A^\dagger \\ A & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.13)$$

Then

$$H_- = A^\dagger A = -\partial_x^2 + W^2 - W', \quad H_+ = AA^\dagger = -\partial_x^2 + W^2 + W'. \quad (2.12)$$

The adjoint and the products refer to their operator domains. For a closed densely defined A , both products are self-adjoint and nonnegative, with

$$\mathcal{D}(A^\dagger A) = \{\psi \in \mathcal{D}(A) : A\psi \in \mathcal{D}(A^\dagger)\}. \quad (2.A7)$$

This is the operator statement behind the formal multiplication of differential expressions [27].

The positive spectra pair through the intertwiner:

$$H_- \psi = E\psi \implies H_+(A\psi) = E(A\psi). \quad (2.16)$$

For a normalised state with $E > 0$, the normalised partner is $E^{-1/2}A\psi$. A zero mode satisfies one first-order equation,

$$\psi_{-,0} = C_- e^{-\int W dx}, \quad \psi_{+,0} = C_+ e^{+\int W dx}. \quad (2.15)$$

Whether either belongs to the Hilbert space is decided by the endpoints and the domain. On the usual full line at most one of the reciprocal solutions is square integrable. Other endpoint settings are handled by the same domain test.

This is the old factorisation method dressed in a useful grading [28, 29]. The grading makes the pairing particularly transparent; the first-order factor supplies the actual computational work. Shape invariance adds an efficient rule for repeating it [30, 31].

2.2.1 The delta as a SUSY example

For $\kappa > 0$ take

$$W(x) = \kappa \operatorname{sgn} x, \quad W' = 2\kappa\delta(x). \quad (2.17)$$

The partners are

$$H_- = -\partial_x^2 + \kappa^2 - 2\kappa\delta(x), \quad H_+ = -\partial_x^2 + \kappa^2 + 2\kappa\delta(x). \quad (2.18)$$

The attractive delta has the zero mode $\sqrt{\kappa}e^{-\kappa|x|}$ in this shifted energy convention. The repulsive partner begins at the common continuum threshold κ^2 . Subtract that constant and the familiar bound energy is $-\kappa^2$.

One can obtain the jump condition from the product domain alone. Let $\mathcal{D}(A) = H^1(\mathbb{R})$, so ψ is continuous. Membership of $A\psi$ in the adjoint domain requires

$$\psi'(0+) + \kappa\psi(0) = \psi'(0-) - \kappa\psi(0), \quad (2.20)$$

hence, for H_- ,

$$[\psi] = -2\kappa\psi(0). \quad (2.21)$$

The other product gives the positive jump. No choice of a value for $\operatorname{sgn} 0$ enters this derivation.

The example has a pleasing economy: supersymmetry changes the sign of the contact coupling and removes the one bound level. The associated scale is already present in κ .

2.2.2 What the intertwiner does to scattering

Assume $W(x) \rightarrow w_L, w_R$ at the two ends and $w_L^2 = w_R^2$, so the partners have the same asymptotic threshold. Applying A to a plane wave multiplies it by $\pm ik + w_{L,R}$. Renormalising the incident wave after that operation gives

$$\begin{aligned} t_{L,+} &= t_{L,-} \frac{ik + w_R}{ik + w_L}, & r_{L,+} &= r_{L,-} \frac{-ik + w_L}{ik + w_L}, \\ t_{R,+} &= t_{R,-} \frac{-ik + w_L}{-ik + w_R}, & r_{R,+} &= r_{R,-} \frac{ik + w_R}{-ik + w_R}. \end{aligned} \quad (2.A8)$$

This is the scattering version of the intertwining relation. It determines the added or removed Jost factor and keeps the asymptotic phases under control [32, 33, 18].

If W is odd, the partners are even and A changes parity. Let $w = W(+\infty)$ and $\alpha(k) = \arctan(w/k)$. The phase relations are

$$\delta_{e,+} = \delta_{o,-} - \alpha \pmod{\pi}, \quad (2.22)$$

$$\delta_{o,+} = \delta_{e,-} - \alpha \pmod{\pi}. \quad (2.23)$$

Both parities are exchanged. When $w = 0$, the operation exchanges the two phase shifts directly. Equivalently, transmission stays fixed and reflection changes sign. The family equation (1.91) is an explicit example.

2.3 Building spectra with SUSY ladders

A single factorisation relates two spectra. A useful ladder begins when the partner can be recognised as another member of the same family, with shifted parameters and a constant energy offset:

$$H_+(a) = H_-(a_1) + R(a). \quad (2.A9)$$

The zero mode is known from its first-order equation. The rest of the spectrum follows from the next rung. Repeating the operation gives sums of the offsets R and products of first-order intertwiners for the wave functions.

In the downward direction the lowest state tells us the superpotential, $W = -\partial_x \log \psi_0$. In the upward direction a suitable seed must be chosen. The seed's zeros and endpoint behaviour decide whether the next Hamiltonian is regular on the desired domain. This is where the innocent-looking algebra has to report back to analysis.

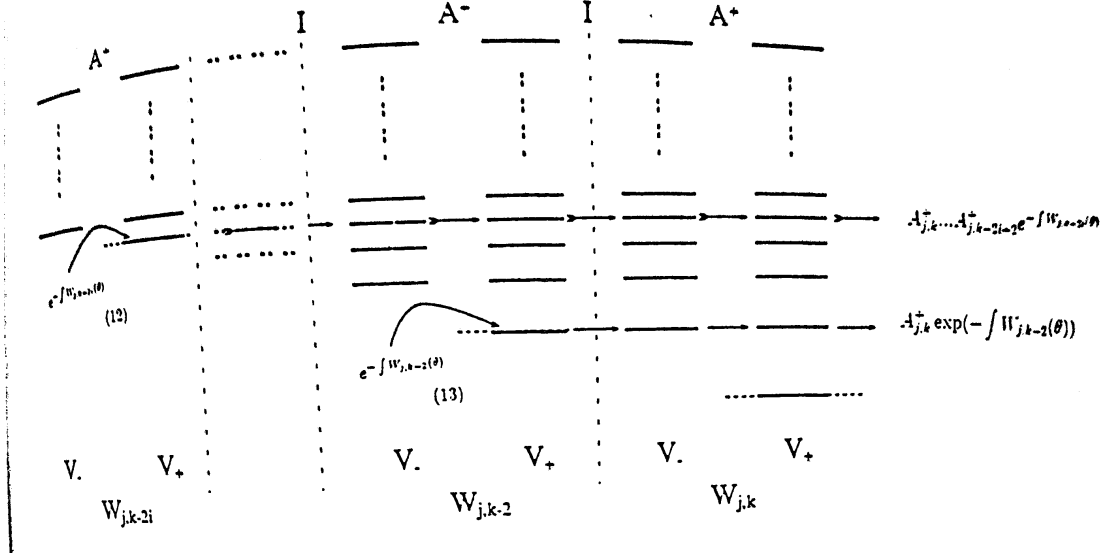


Figure 2.1: Joining successive SUSY pairs into a ladder, restored from original Figure 2.1. Each stage adds a lowest level and carries the previously calculated wave functions through the adjoint intertwiner, denoted by A^+ in the drawing. The identity steps join consecutive partners after their energy origins have been aligned.

2.3.1 Five elementary families attached to free motion

Start by solving the Riccati equation $W^2 \pm W' = \text{constant}$. Up to shifts, rescalings and overall sign choices, its nonconstant elementary branches are

$$\frac{1}{x}, \quad \tan x, \quad \tanh x, \quad \cot x, \quad \coth x. \quad (2.27)$$

Constant solutions supply trivial free partners. Iteration of the nonconstant branches gives the familiar potentials

$$\frac{l(l+1)}{x^2}, \quad l(l+1) \sec^2 x, \quad -l(l+1) \operatorname{sech}^2 x, \quad l(l+1) \csc^2 x, \quad l(l+1) \operatorname{csch}^2 x, \quad (2.28)$$

with the appropriate additive constants and domains.

The circular branches have periodically spaced singular endpoints and naturally define compact interval problems. The $1/x$ and $\coth x$ branches have a singularity at the origin and naturally define radial half-line problems. Their full-line realisations require an explicit choice of interface domain. The $\tanh x$ branch is regular on the whole line and gives the transparent ladder.

For

$$V_l(x) = -l(l+1) \operatorname{sech}^2 x, \quad l = 0, 1, 2, \dots, \quad (2.29)$$

one obtains

$$r_l(k) = 0, \quad t_l(k) = \prod_{j=1}^l \frac{k + ij}{k - ij}. \quad (2.30)$$

The high-energy limit is one. The poles $k = ij$ correspond to energies $-j^2$. A bounded zero-energy solution is also present; its parity alternates with l . It remains a threshold resonance on the full line. The general noninteger coupling in Chapter 3 shows how the reflectionless values sit inside the continuous family.

(2026) Crum's Wronskian construction extends these successive transformations to arbitrary admissible seed energies [29]. The Kay–Moses determinant in equation (1.A13) is the inverse-scattering counterpart. The regular sech^2 ladder is particularly simple because its poles fall at evenly spaced imaginary momenta.

2.3.2 The geometric superpotentials

The three geometries now fit directly into the factorisation:

$$W_h(r) = \frac{p}{2} \coth r + q \coth 2r, \quad (2.31)$$

$$W_c(r) = \frac{p}{2} \cot r + q \cot 2r, \quad (2.32)$$

$$W_f(r) = \frac{n-1}{2r}. \quad (2.33)$$

The compact and noncompact expressions are related by analytic continuation of their radial coordinate and spectral parameters. The flat case is the small-curvature contraction. The inverse-square term scales in the same way as the kinetic energy, explaining its recurrence in the scale arguments of the last chapter.

For a sphere the geometric ground state in flat radial measure is

$$u_0(r) = (\sin r)^{(n-1)/2}. \quad (2.24)$$

Undoing the unitary substitution returns the constant radial function on S^n . A lowest Schrödinger wave function with a visible sine power can therefore represent the simplest possible function on the original manifold. The measure is doing precisely the work we asked it to do.

2.3.3 Compact spectra and multiplicity shifts

To calculate the compact spectrum, use the superpotential with the sign convenient for a lowest state,

$$w_{A,B}(r) = -A \cot r + B \tan r, \quad u_0 = (\sin r)^A (\cos r)^B. \quad (2.34)$$

Then

$$H_-(A, B) = -\partial_r^2 + A(A-1) \csc^2 r + B(B-1) \sec^2 r - (A+B)^2, \quad (2.36)$$

and

$$H_+(A, B) = H_-(A+1, B+1) + 4(A+B+1). \quad (2.A10)$$

The shifts can be composed with the parameter reflections that preserve the inverse-square coefficients. This produces the multiplicity lattice of the original construction. The simple positive-state ladder above gives the energy sum without having to stop at the isospectral intermediate pairs.

To see the two elementary directions, temporarily remove the constant term and write

$$L_{A,B} = -\partial_r^2 + A(A-1)\csc^2 r + B(B-1)\sec^2 r. \quad (2.A11)$$

Its coefficients are invariant under $A \mapsto 1-A$ and $B \mapsto 1-B$. Factoring with $w_{A,B}$ gives the edge $(A, B) \mapsto (A+1, B+1)$. Factoring with $w_{A,1-B}$ and using the second reflection gives $(A, B) \mapsto (A+1, B-1)$. In root multiplicities these read

$$(p, q) \mapsto (p, q+2), \quad (p, q) \mapsto (p+4, q-2). \quad (2.A12)$$

The second keeps ρ fixed. It is the useful isospectral direction when the endpoint branches carried by the intertwiner agree with the chosen geometric domains [26]. The first advances the ordinary state-deleting ladder. The multiplicities make the geometry visible, while the endpoint test decides which arrows compare the desired operators.

The full parameter-plane drawing also records three composed lowering routes,

$$(p, q) \mapsto \begin{cases} (p, q-2), \\ -(p+2q-2), q-2, \\ (p+2q-2, -q). \end{cases} \quad (2.A13)$$

In (A, B) these targets are $(A-1, B-1)$, $(-A, B-1)$ and $(A-1, -B)$. They arise by joining SUSY steps with the parameter reflections. The principal route decreases q at fixed p ; its reverse is the positive-state ladder used above. The alternative routes pass through reflected branches and intermediate isospectral pairs. The superpotential can change sign or acquire a nonnormalisable zero solution during such a passage, while the partner differential expression remains calculable. Energy offsets and endpoint domains are carried through each identification.

Summing the offsets gives

$$E_\nu = 4\nu(\nu + A + B) = 4\nu(\nu + \rho), \quad \nu = 0, 1, 2, \dots \quad (2.38)$$

The transformed eigenfunctions are

$$u_\nu(r) = C_\nu(\sin r)^A(\cos r)^B P_\nu^{(A-1/2, B-1/2)}(\cos 2r). \quad (2.A14)$$

After dividing by $J_c^{1/2}$, the radial functions are Jacobi polynomials. Orthogonality follows in the original radial measure or, equivalently, in flat measure after the unitary transformation.

Compact space	p	q	Isotropic eigenvalues
$\mathbb{R}P^n$	$n-1$	0	$4\nu(\nu + (n-1)/2)$
$\mathbb{C}P^n$	$2n-2$	1	$4\nu(\nu + n)$
$\mathbb{H}P^n$	$4n-4$	3	$4\nu(\nu + 2n+1)$
Cayley plane $\mathbb{O}P^2$	8	7	$4\nu(\nu + 11)$

Table 2.2: Compact radial spectra. Real projective space has the locally unit-sphere metric and selects even spherical degrees. The other spaces use the standard rank-one scale with radial interval $[0, \pi/2]$. A common rescaling of the metric rescales every displayed energy.

For the unit sphere itself, the interval reaches π and the spectrum is

$$E_{l,n} = l(l+n-1), \quad l = 0, 1, 2, \dots \quad (2.39)$$

For $n \geq 2$, the isotropic eigenfunctions are Gegenbauer polynomials $C_l^{(n-1)/2}(\cos r)$; on S^1 they are $\cos(lr)$. On S^2 the first two examples are

$$\psi_0(r) = 1, \quad E_0 = 0, \quad (2.41)$$

$$\psi_1(r) = \cos r, \quad E_1 = 2. \quad (2.42)$$

In flat radial measure they become $\sqrt{\sin r}$ and $\sqrt{\sin r} \cos r$. Those two familiar functions are a useful sanity check on the ladder.

(2026-gpt) The metric normalisation in [table 2.2](#) makes the factor of four explicit. Restricting unit-sphere harmonics to even degrees $l = 2\nu$ gives $4\nu(\nu + (n-1)/2)$ on $\mathbb{R}P^n$. Writing $\nu(\nu + \rho)$ instead corresponds to a metric whose lengths have been multiplied by two. Both conventions are useful; here the unit-sphere convention fixes the comparison.

At fixed ρ , [equation \(2.38\)](#) gives the same sequence of isotropic eigenvalues. The endpoint exponents and radial measures still depend on A and B . Full Laplace spectra also include angular sectors and degeneracies. The equality stated here concerns the one-dimensional isotropic sequence, which is exactly what the reduction calculates.

2.A1 The larger hypergeometric family (2026)

The multiplicity lattice contains many points with no compact homogeneous space attached to them. It is useful to let the parameters move freely and keep the differential problem. This larger family has a well-developed mathematical home in hypergeometric functions associated with root systems and their shift operators [\[34, 35\]](#).

For the noncompact rank-one case put

$$\alpha = \frac{p+q-1}{2}, \quad \beta = \frac{q-1}{2}, \quad \rho = \alpha + \beta + 1. \quad (2.A15)$$

The radial function normalised by $\phi_k(0) = 1$ is

$$\phi_k(r) = {}_2F_1\left(\frac{\rho - ik}{2}, \frac{\rho + ik}{2}; \alpha + 1; -\sinh^2 r\right). \quad (2.A16)$$

Its asymptotic decomposition is

$$\phi_k(r) \sim c(k)e^{(ik-\rho)r} + c(-k)e^{(-ik-\rho)r}, \quad (2.A17)$$

where, in this normalisation,

$$c(k) = \frac{2^{\rho-ik}\Gamma(\alpha+1)\Gamma(ik)}{\Gamma((\rho+ik)/2)\Gamma((\alpha-\beta+1+ik)/2)}. \quad (2.A18)$$

The outgoing-to-incoming ratio in flat radial measure is $c(k)/c(-k)$. A physical half-line phase is obtained after fixing the comparison endpoint convention, just as in [Chapter 1](#). Multiplicity shifts become gamma-function recurrence identities for this ratio. The scattering and ladder descriptions are two views of the same hypergeometric structure.

The rank-one geometric families occupy distinguished integer multiplicities. Analytic continuation in those parameters gives a larger set of exactly solvable radial equations. Endpoint conditions and square integrability remain part of the specification, especially where an inverse-square coefficient lies in the limit-circle range.

2.A1.1 Further shape-invariant families (2026)

The five elementary Riccati branches are a good entry point. Iterated Darboux transformations with more general seeds give rational deformations and exceptional orthogonal polynomials, with further shape-invariant systems [36]. The polynomial factor changes, while the intertwining and domain questions remain recognisable. This extends the working principle of the original ladder well beyond the elementary trigonometric and hyperbolic catalogue.

2.A1.2 Odd-dimensional de Sitter transparency (2026)

Lagogiannis, Maloney and Wang reduce free-field mode equations in global de Sitter space to one-dimensional Pöschl–Teller problems [37]. In odd spacetime dimension the relevant potentials fall on reflectionless ladders. The vanishing of the reflection coefficient translates into vanishing in/out particle production in their setting.

The spatial Schrödinger problem in this chapter and the time-dependent mode equation have different physical interpretations, but the same factorisation calculates the relevant connection coefficients. This is a particularly direct later meeting of geometry, dimensional dependence and SUSY transparency.

2.A1.3 Conformal blocks and optical structures (2026)

The correspondence between scalar conformal blocks and Calogero–Sutherland eigenfunctions uses coupled Pöschl–Teller systems and Heckman–Opdam functions [38, 39]. The root multiplicities become continuous parameters of conformal data. The radial equations and their shift relations therefore enter a modern field-theoretic calculation through a very concrete differential operator.

In a different direction, supersymmetric optical design uses intertwiners to construct structures with controlled relations between their mode spectra and scattering amplitudes [40]. There too the asymptotic multiplication factors in [equation \(2.A8\)](#) have a direct interpretation. A calculation that begins as a way of simplifying a quantum spectrum can finish as a way of designing a waveguide.

The geometric chapter thus closes on the same practical point with which it began: factorisation makes a complicated comparison calculable. Its success depends on carrying the measure, the endpoint conditions and the asymptotic normalisation alongside the algebra.

Chapter 3

Trajectories and poleology

*An algebraist you became,
refined to the very essence,
an oligarch of science,
a mathematical grandee.*

*You once begged for solutions
to a miserable equation;
now you inspect the postulates
and scrutinise each definition.*

*Yet everyone remembers
your matrices and their roots,
and how the method of Sturm
kept you in perfectly good spirits.*

E. Gentile, *The Algebraist*, tango; translated selection

Coupling and range. Bound-state trajectories. Regular minima and Coulomb singularities. Phase shifts. Square wells, one delta, two deltas and three. Lambert branches and resonance coalescences. Transparent potentials and the poles hidden by exact cancellation.

3.1 Eigenvalue trajectories

Bound states lie at poles $k = i\kappa$, $\kappa > 0$, with energy $E = -\kappa^2$. Before chasing poles around the whole complex plane, it is sensible to watch these particular ones. Vary the coupling, calculate the energy levels, and draw the resulting curves. The spirit resembles a Regge plot, with the coupling taking the role of the continued parameter. The analytic parameter in the present calculation is the coupling itself.

Take

$$H(g, b) = -\partial_x^2 + gU(x/b). \quad (3.1)$$

Rescale $x = by$. The dimensionless problem is

$$b^2 H = -\partial_y^2 + \lambda^2 U(y), \quad \lambda = b\sqrt{g}. \quad (3.2)$$

Therefore

$$\kappa_n(g, b) = b^{-1} F_n(gb^2), \quad E_n(g, b) = b^{-2} \mathcal{E}_n(gb^2). \quad (3.3)$$

Changing the range or changing the coupling follows the same dimensionless curves, with a final rescaling of the units. For a delta coefficient the dimensionless combination is instead gb , since that coefficient has dimension L^{-1} .

For the regular wells we hold the range at one and plot $\kappa_n = \sqrt{-E_n}$ against $\sqrt{g}$. This explains the nearly straight lines in the old figures: the plotted vertical variable is inverse length. The energy itself has a leading term proportional to g .

3.1.1 A selection of potentials

The regular examples are

$$\begin{aligned} U_{\text{sq}}(x) &= -\mathbf{1}_{|x|<1}, & U_{\text{exp}}(x) &= -e^{-|x|}, \\ U_G(x) &= -\frac{1}{2}e^{-x^2/2}, & U_{\text{PT}}(x) &= -\text{sech}^2 x. \end{aligned} \quad (3.A1)$$

Their spectra can be checked through matching, Bessel functions, numerical integration and the exact Pöschl–Teller formulas. Using several solvable cases around the numerical one is a comfortable arrangement: a computer has fewer opportunities to persuade us that a wrong sign is a physical effect.

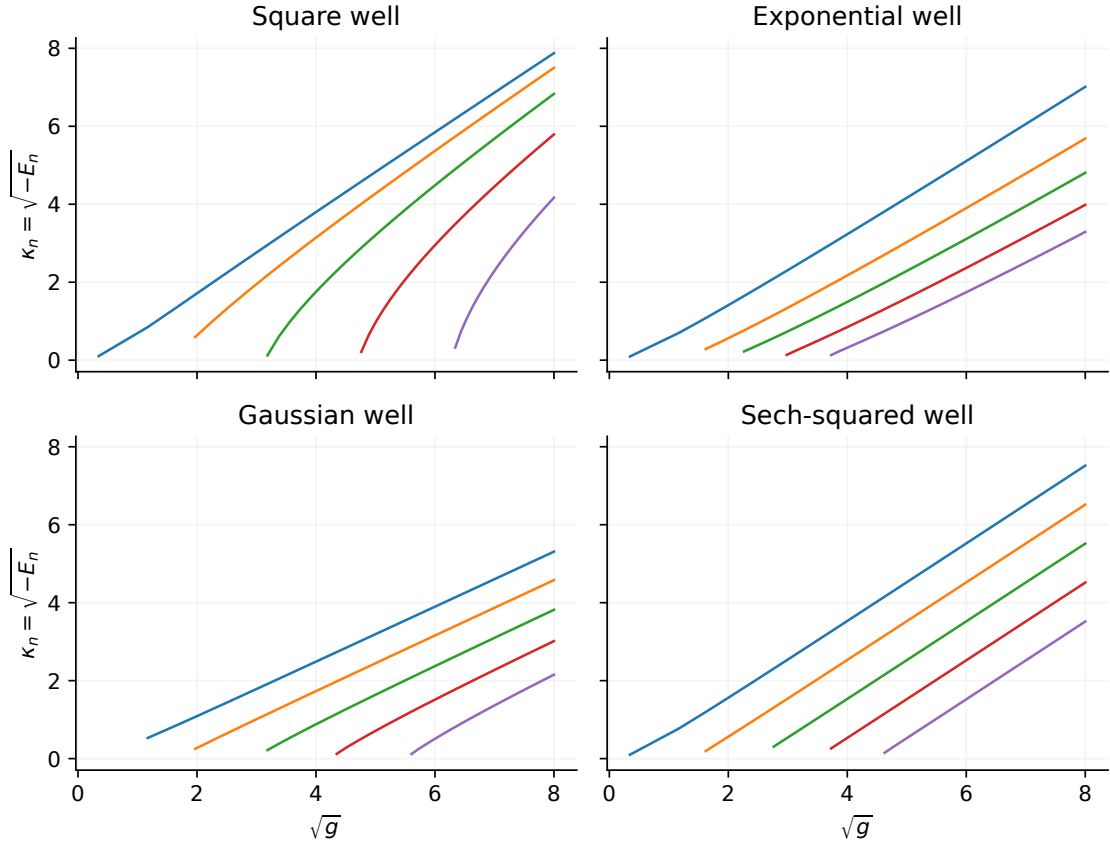


Figure 3.1: Inverse binding lengths against $\sqrt{g}$ for the four regular wells. **(2026)** Recalculated with full-line finite differences on $[-16, 16]$, spacing $1/75$, with Dirichlet endpoints. Levels very close to zero are suppressed to reduce finite-box threshold artefacts. The exact sech^2 spectrum provides an independent discretisation check.

The singular comparisons use the half-line operators

$$U_Y(r) = -\frac{e^{-r}}{r}, \quad U_H(r) = -\frac{1}{e^r - 1}, \quad U_C(r) = -\frac{1}{r}, \quad (3.A2)$$

with the Dirichlet/Friedrichs condition at zero. They describe the odd sector of the corresponding symmetric full-line construction. That explicit endpoint choice keeps the Coulomb

singularity under control. Other extensions of singular full-line expressions amount to additional boundary data.

The Coulomb and Hulthén inverse lengths are especially simple:

$$\kappa_{C,n} = \frac{g}{2n}, \quad \kappa_{H,n} = \frac{g - n^2}{2n}, \quad g > n^2, \quad n = 1, 2, \dots \quad (3.A3)$$

The Yukawa equation supplies the numerical comparison. The growth of κ is now proportional to g , so it looks quadratic on the horizontal variable $\sqrt{g}$. The rapidly screened and unscreened examples share their leading Coulomb scale while retaining distinct subleading terms and thresholds.

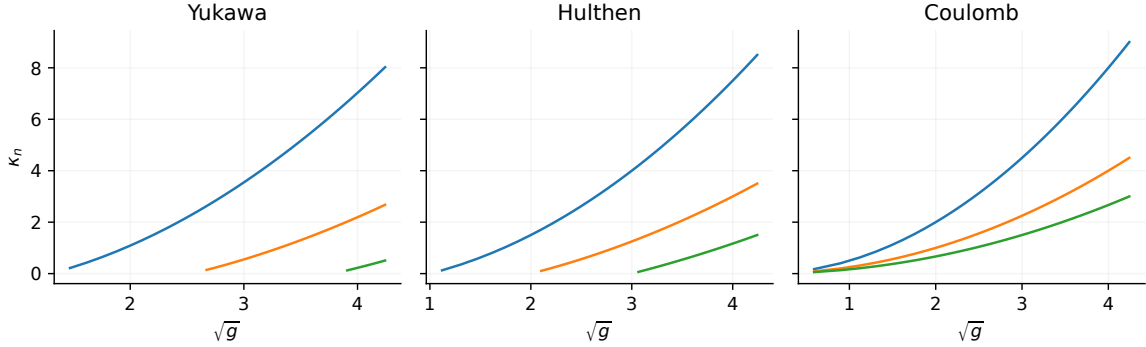


Figure 3.2: The first three half-line inverse binding lengths for Yukawa, Hulthén and Coulomb potentials. **(2026)** The last two panels use equation (3.A3); Yukawa uses spacing 0.0025 on $(0, 30)$ with Dirichlet endpoints. The plotted ordinate and coupling convention are the same in all three panels.

For a regular integrable attractive potential gU with $\int U < 0$, the weak-coupling ground state has

$$E_0(g) = -\frac{g^2}{4} \left(\int_{\mathbb{R}} U(x) dx \right)^2 + o(g^2), \quad g \downarrow 0, \quad (3.A4)$$

under the usual moment assumptions [41]. Excited states require finite thresholds. This low-coupling behaviour and the strongly localised regime belong to different parts of the same curves.

3.1.2 The large-coupling calculation

For an isolated normalised eigenstate, Hellmann–Feynman gives

$$\frac{dE_n}{dg} = \langle \psi_n(g), U \psi_n(g) \rangle. \quad (3.6)$$

For an isolated level, the corresponding perturbation expansion reads

$$E_n(g + \delta g) = E_n(g) + \delta g \langle \psi_n, U \psi_n \rangle + O(\delta g^2). \quad (3.4)$$

With $R_n^\perp = (E_n - H)^{-1}$ on the orthogonal complement of ψ_n , its second-order term is

$$(\delta g)^2 \langle U \psi_n, R_n^\perp U \psi_n \rangle. \quad (3.5)$$

The eigenfunction depends on g . Its derivative contains the other discrete states and the continuum through the reduced resolvent. For a discrete basis one recognises the familiar part

$$\partial_g \psi_n = \sum_{m \neq n} \frac{\langle \psi_m, U \psi_n \rangle}{E_n - E_m} \psi_m + \text{continuum contribution}, \quad (3.A5)$$

after fixing the phase by $\langle \psi_n, \partial_g \psi_n \rangle = 0$. This formula is useful locally in g ; the actual strong-coupling estimate follows more directly from localisation.

(2026) The strong-coupling expansion of Galindo and Pascual and the semiclassical analysis of Simon place this localisation argument on its natural footing [118, 42]. Suppose a regular minimum is at zero and

$$U(x) = -v_0 + c|x|^p + o(|x|^p), \quad v_0, c > 0. \quad (3.A6)$$

Set $x = (gc)^{-1/(p+2)}y$. For a fixed low-lying index, the leading local problem is

$$H + gv_0 = (gc)^{2/(p+2)}(-\partial_y^2 + |y|^p) + \dots, \quad (3.A7)$$

so

$$E_n(g) = -gv_0 + (gc)^{2/(p+2)}e_{n,p} + o(g^{2/(p+2)}). \quad (3.A8)$$

Here $e_{n,p}$ is the spectrum of the limiting local operator. This is the semiclassical organisation of the strong-coupling problem [42]. It explains both the common leading slope and the dependence on the shape of the bottom.

For a smooth quadratic minimum, $p = 2$,

$$E_n = -gv_0 + (2n+1)\sqrt{gc} + o(\sqrt{g}), \quad \kappa_n = \sqrt{gv_0} - (n + \tfrac{1}{2})\sqrt{c/v_0} + o(1). \quad (3.A9)$$

The inverse-length lines have constant offsets. For $U = -\text{sech}^2 x$, the exact result is

$$s = \frac{\sqrt{1+4g}-1}{2}, \quad E_n = -(s-n)^2, \quad 0 \leq n < s, \quad (3.A10)$$

and $\kappa_n = \sqrt{g} - n - 1/2 + O(g^{-1/2})$, in agreement with the local harmonic oscillator.

The exponential well has a cusp, $p = 1$. Its local correction is $g^{2/3}$ and is governed by Airy eigenvalues: zeros of Ai' for the even sector and of Ai for the odd one. Consequently the correction to $\kappa \sim \sqrt{g}$ grows as $g^{1/6}$. The apparent straightness over a finite plotting range has a calculable curvature beneath it.

A flat square-well bottom gives a third case. For fixed n and large depth,

$$E_n = -g + \left(\frac{(n+1)\pi}{2b} \right)^2 + O(g^{-1/2}), \quad (3.A11)$$

with the appropriate fixed width $2b$. The correction to κ decreases as $g^{-1/2}$. Thus the smooth, cusp and flat cases share their leading depth while the next term records the local minimum.

For a Coulomb singularity $-g/r$, the natural length is g^{-1} and the energy scale is g^2 . Expanding the screening at the origin gives, for the Yukawa example,

$$-\frac{ge^{-r}}{r} = -\frac{g}{r} + g - \frac{g}{2}r + \dots. \quad (3.8)$$

At fixed principal index n ,

$$E_{Y,n} = -\frac{g^2}{4n^2} + g - \frac{3}{2}n^2 + O(g^{-1}). \quad (3.9)$$

The displayed constant follows from the Coulomb expectation $\langle r \rangle = 3n^2/g$ in our units. The Hulthén formula gives exactly

$$E_{H,n} = -\frac{g^2}{4n^2} + \frac{g}{2} - \frac{n^2}{4}. \quad (3.A12)$$

The comparison is now quantitative: the singularity fixes the leading scale, and the short-distance screening expansion fixes the next terms.

3.1.3 Following the phase shifts

The discrete spectrum follows the positive imaginary axis. The physical scattering amplitudes live on the positive real axis. Parity lets us inspect the latter through two phase shifts, so the next natural plot keeps both channels visible.

For $V = -s(s+1) \operatorname{sech}^2 x$, the exact formulas below give the curves in figure 3.3. At large k the phases vary smoothly and tend to zero in the chosen branches. At small k their evolution records the appearance of new levels. Integer s gives the transparent values: $r = 0$ and the even and odd phase shifts agree modulo π . The threshold resonance alternates between the two parities as s passes through successive integers.

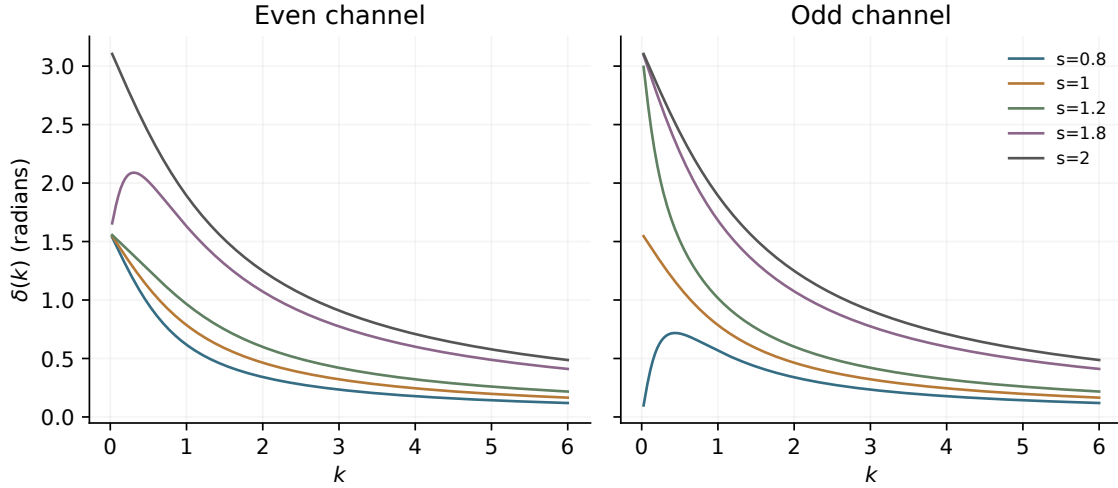


Figure 3.3: Parity phase shifts for $V = -s(s+1) \operatorname{sech}^2 x$, calculated from the gamma-function amplitudes. **(2026)** The phase branches are unwrapped from high momentum, where they tend to zero. Integer and nearby noninteger values show the threshold rearrangement.

3.2 Poleology of the scattering matrix

The real potential symmetry pairs continued poles at k and $-\bar{k}$. Upper imaginary poles associated with decaying Jost solutions are bound states. Negative imaginary poles are virtual or antibound states. Poles with nonzero real part in the lower half-plane describe outgoing resonances, with complex energy $E = k^2$.

As a real coupling varies, a symmetric resonance pair can meet on the negative imaginary axis and split into two imaginary poles. One branch can then rise to threshold and become a bound state. This is a common and visually memorable mechanism. The detailed motion is determined by the Jost function and includes possible cancellations, threshold zeros and excursions to infinity. The analytic equations are the guide when a plotted trajectory becomes ambiguous [43, 44, 45, 46].

(2026) Analytic deformation gives a spectral formulation of outgoing resonances [47, 105]. A generic coalescence is a double zero of a scalar Jost function,

$$J(k_*, g_*) = J_k(k_*, g_*) = 0, \quad J_{kk}(k_*, g_*) J_g(k_*, g_*) \neq 0. \quad (3.A13)$$

(2026-gpt) Expanding the Jost function to second order gives the local form

$$k - k_* = \pm \sqrt{-\frac{2J_g}{J_{kk}}(g - g_*)} + O(g - g_*). \quad (3.A14)$$

For an uncanceled scattering pole, the outgoing resonance operator has the associated Jordan-chain description [48]. This is the exceptional-point language appropriate to the pole problem. The ordinary self-adjoint bound-state operator retains its usual spectral properties.

3.2.1 The square well and barrier

Take $V(x) = V_0$ for $|x| < b$ and zero outside. Let

$$V(x) = V_0 \mathbf{1}_{|x| < b}, \quad q^2 = k^2 - V_0. \quad (3.10)$$

Matching at b , with free-normalised parity Jost functions, gives

$$\begin{aligned} J_e(k) &= e^{ikb} \left[\cos qb - i \frac{q}{k} \sin qb \right], \\ J_o(k) &= e^{ikb} \left[\cos qb - i \frac{k}{q} \sin qb \right]. \end{aligned} \quad (3.15)$$

The functions $\cos qb$, $\sin qb/q$ and $q \sin qb$ are entire in q^2 , so changing the square-root branch of q leaves the physical expressions unchanged. The even amplitude is

$$S_e(k) = e^{-2ikb} \frac{k + iq \tan qb}{k - iq \tan qb}, \quad (3.14)$$

understood through the un-divided sine and cosine form at zeros of a denominator used as a coordinate.

For a bound state $k = i\kappa$, the familiar matching equations are

$$q \tan qb = \kappa \quad (e), \quad -q \cot qb = \kappa \quad (o), \quad q^2 = -V_0 - \kappa^2. \quad (3.A15)$$

The even lowest state enters as soon as the well becomes attractive. Higher levels appear at the finite couplings at which one of the zero-energy matching conditions is met.

The lower-half-plane poles have a regular pattern. In the weak-potential limit the nontrivial resonance roots retreat far downwards. As the well deepens, pairs return towards the negative imaginary axis and coalesce. For a generic square-well coalescence at nonzero coupling, differentiation of the parity denominator gives

$$k_* = -\frac{i}{b}. \quad (3.A16)$$

The coupling values differ from pair to pair. The common imaginary position is a property of this particular sharp-edged potential. Keeping b explicit resolves the numerical factors that otherwise depend on whether the width was called one or two.

The common height $k_* = -2i$ in the photographed original Figures 3.5 and 3.6 corresponds to $b = 1/2$ in our convention. The present plots use $b = 1$, giving $k_* = -i$. The two drawings agree after this length rescaling. Their arrows follow the changing real coupling; a picture of the complete root set and an oriented trajectory provide complementary information about the same Jost equation.

3.2.2 One delta in motion

For $V = g\delta(x)$, equation (1.A9) gives one pole,

$$k(g) = -\frac{ig}{2}. \quad (3.A17)$$

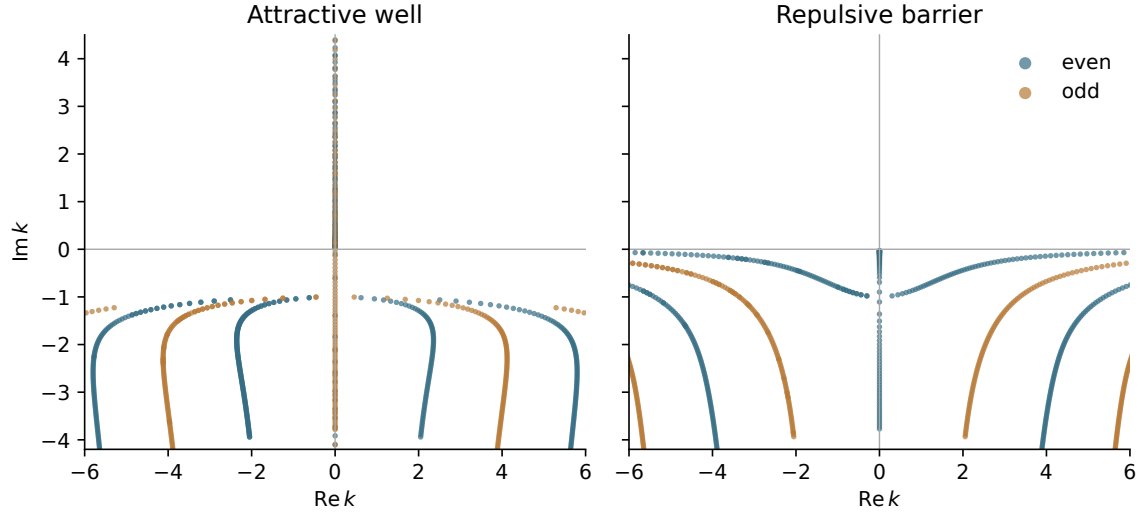


Figure 3.4: Square-well and square-barrier pole maps at half-width $b = 1$. **(2026)** Each point is obtained from a parity Jost equation at a real coupling; blue and orange denote even and odd channels. The imaginary-axis coalescences of the well accumulate at the common height $\text{Im } k = -1$. The attractive and repulsive panels use the same momentum units.

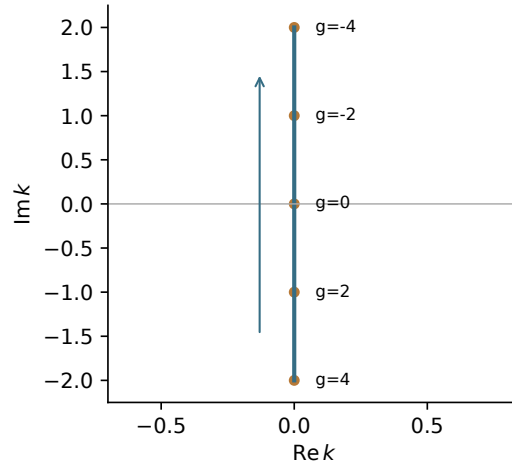


Figure 3.5: The single delta pole, $k = -ig/2$. Its trajectory is the simplest example of a virtual state passing through threshold into a bound state.

It moves straight up the imaginary axis as g decreases. It is virtual for $g > 0$, reaches zero at $g = 0$, and becomes the bound state for $g < 0$. A one-dimensional attractive contact produces its first level with arbitrarily small attraction. This explains why the pole can pass through threshold exactly where the coupling changes sign.

The example gives a scattering matrix controlled by one physical pole and a simple rational background. The transparent ladder provides another rational example with several bound poles. Finite-range wells supply a contrasting infinite set of resonance roots.

3.2.3 Two and three deltas

Return to equations (1.55) and (1.57). The even zero equation is

$$k + \frac{ig}{2}(1 + e^{2ika}) = 0. \quad (3.A18)$$

The odd equation has a minus sign before the exponential. Multiplying an equation by k introduces a zero at the origin, so the $k = 0$ case is checked against the original Jost function.

For large $|g|$ the finite-momentum resonance roots approach the cavity values

$$k_e \longrightarrow \frac{(2n+1)\pi}{2a}, \quad (3.22)$$

$$k_o \longrightarrow \frac{n\pi}{a}. \quad (3.23)$$

Their widths tend to zero. The limiting impenetrable walls isolate the interval $(-a, a)$ and its positive discrete eigenvalues. For every finite real scalar coupling, the corresponding positive-energy states remain resonances of the coupled line. Strong attractive couplings also produce deeply negative states localised at the centres.

(2026-gpt) The double-delta trajectories have an exact expression in terms of the Lambert function:

$$\begin{aligned} k_{e,n}(g) &= \frac{i}{2a} \{W_n(-age^{ag}) - ag\}, \\ k_{o,n}(g) &= \frac{i}{2a} \{W_n(+age^{ag}) - ag\}. \end{aligned} \quad (3.A19)$$

The identically zero roots created by multiplying the odd equation by k are discarded using the original Jost condition. A small positive imaginary part of g fixes continuation across a branch boundary. The multivalued Lambert function packages the same branch choice that the numerical continuation has to make [49].

The first even coalescence occurs at the Lambert branch point $-1/e$. Put

$$w = W_0(e^{-1}) = 0.278464542761 \dots \quad (3.A20)$$

Then

$$ag_* = w, \quad k_* = -\frac{i(1+w)}{2a}. \quad (3.A21)$$

At $a = 1$ this gives $g_* = 0.2784645428$ and $\text{Im } k_* = -0.6392322714$. The odd bound state appears at $g = -1/a$. The length factors in these two statements are now attached to their equations.

Now add a third delta at the origin:

$$V(x) = g\{\delta(x-a) + \delta(x) + \delta(x+a)\}. \quad (3.24)$$

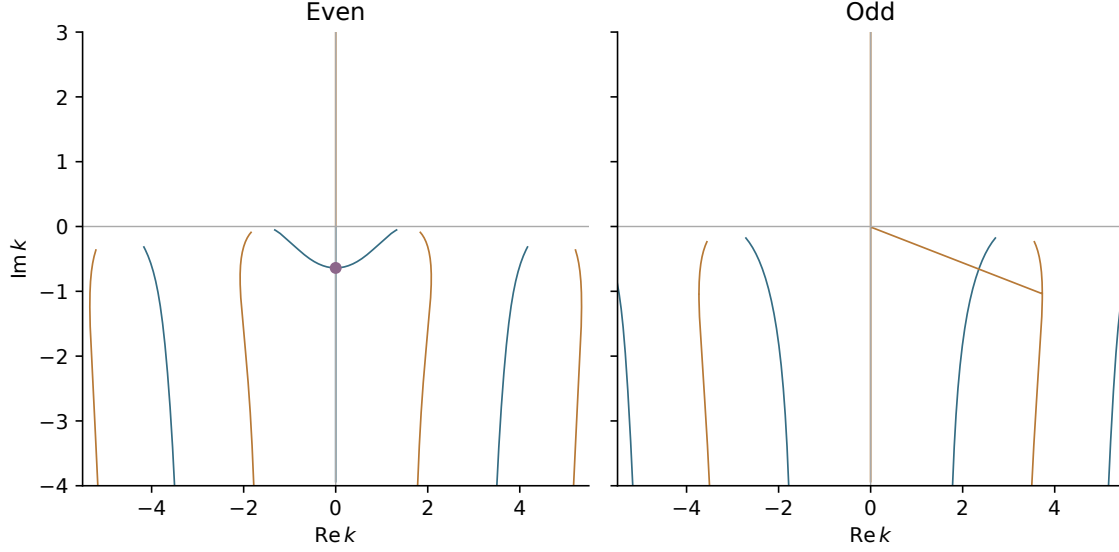


Figure 3.6: Double-delta pole trajectories from [equation \(3.A19\)](#), with $a = 1$. **(2026-gpt)** Blue denotes positive coupling and orange negative coupling. The marked even coalescence is given by [equation \(3.A21\)](#). The odd zero at $k = 0$ introduced by algebraic multiplication is removed.

The odd wave still vanishes at zero, so its Jost function stays exactly the one in [equation \(1.57\)](#). The even endpoint condition changes to $\psi'(0+) = g\psi(0)/2$. Direct propagation gives

$$J_e(k) = 1 + \frac{ig}{k} \left(\frac{1}{2} + e^{ika} \cos ka \right) + \frac{ig^2}{2k^2} e^{ika} \sin ka. \quad (3.26)$$

At large attractive coupling there can be two even bound states and one odd bound state. The first two even coalescences, in the convention $a = 1$, are

	$\kappa_* = -\text{Im } k_*$	g_*
First	0.7350350081	0.2149448494
Second	0.3741675505	-2.9358517759

For another half-separation, both momenta and couplings scale by $1/a$. The two imaginary heights differ. A symmetric three-centre example already carries enough structure to distinguish them, while the square well retains its common touchdown height.

The phrase touchdown is a handy description of the drawing. [Equation \(3.A14\)](#) gives its local analytic content. It also explains the exchange of branches under continuation around the exceptional coupling.

3.2.4 Transparent potentials and exact cancellation

For the Pöschl–Teller well, write

$$V = -s(s+1) \text{sech}^2 x, \quad (3.A22)$$

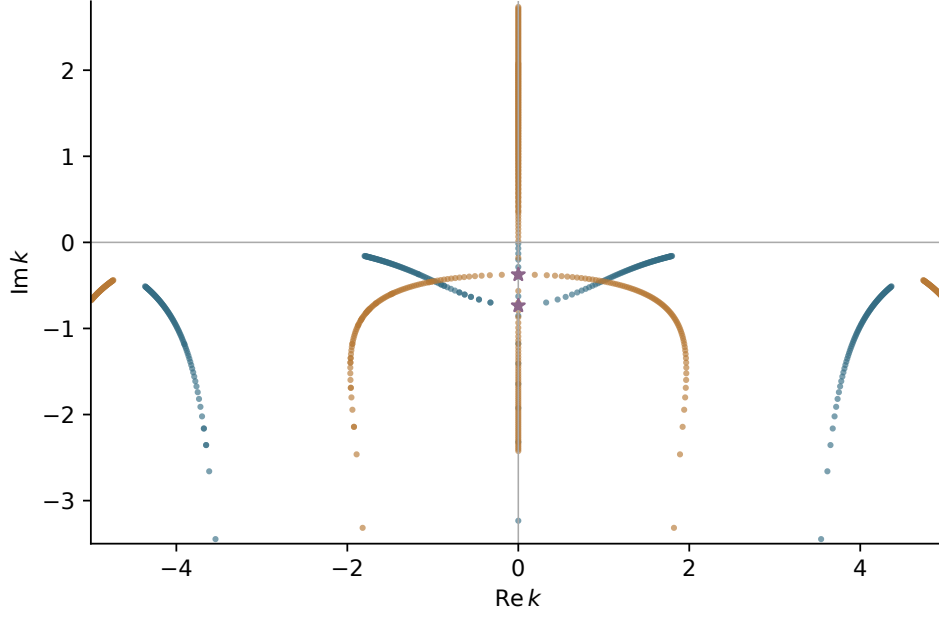


Figure 3.7: Even poles of the triple delta at $a = 1$. **(2026)** The stars mark the two double zeros listed in the text. Positive and negative real couplings are shown in blue and orange. The odd channel coincides with the double-delta odd channel.

with real $s > 0$. Hypergeometric connection formulas give

$$A(k) = \frac{\Gamma(-ik)\Gamma(1-ik)}{\Gamma(-ik-s)\Gamma(1-ik+s)}, \quad (3.29)$$

$$B(k) = \frac{\Gamma(ik)\Gamma(1-ik)}{\Gamma(-s)\Gamma(1+s)} = \frac{i \sin \pi s}{\sinh \pi k}, \quad (3.30)$$

$$t(k) = A(k)^{-1}, \quad r(k) = B(k)A(k)^{-1}. \quad (3.28)$$

The parity amplitudes are $(1 \pm B)/A$. Their physical bound-state poles occur at $k = i(s - n)$ with $s - n > 0$, and their parity alternates with n .

For integer $s = l$, B vanishes and the gamma recurrences reduce t to equation (2.30). The transparent rational result is an exact cancellation at those couplings. Moving s continuously restores reflection and the more elaborate continuation pattern.

The exponential tail also permits singularities of continued reflection or parity amplitudes beyond their initial analytic strip. These include the coupling-independent singularities traditionally called redundant or Ma poles [9, 10, 11]. Bound states are selected by the decaying Jost solution and its endpoint condition. This test keeps the spectral interpretation attached to the Hamiltonian even when an analytically continued ratio contains additional special-function poles.

(2026) The general reflectionless family has independent discrete momenta and norming constants, as in equation (1.A13). Its transmission is still a finite product, while the potential can have a much richer shape. Figure 3.8 compares two norming choices at the same two bound-state momenta. The real-axis amplitudes agree, and the spatial profiles retain the information carried by those norming constants.

This collection has taken us from a pole moving along one straight line to an infinite set of branches, and from a simple bound-state asymptotic to a precise local scaling law. The next chapter changes the question. We keep the support at a single point and ask how much

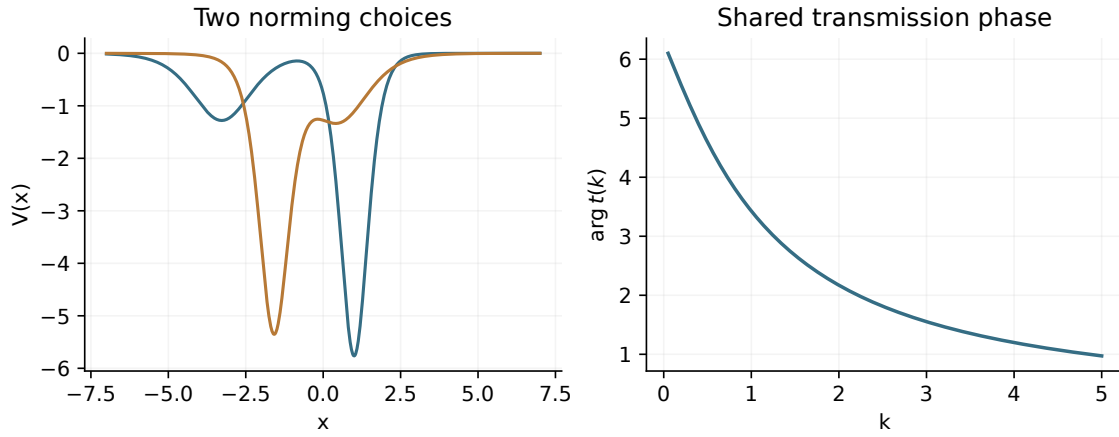


Figure 3.8: Two reflectionless potentials with $\kappa_1 = 0.8$, $\kappa_2 = 1.7$ and different positive norming constants, calculated from [equation \(1.A13\)](#). **(2026)** Their transmission phase is identical. This makes the role of the discrete norming data visible without relying on a translation of a single well.

freedom remains in the operator itself.

Chapter 4

Point-supported interactions

A point is that which has no part.

Euclid, *Elements*

Probability conservation and self-adjoint extensions. Boundary matrices and scattering. Delta and discontinuity contacts. Regularisation by dipoles, resonant profiles and several layers. Distributions on split test spaces. SUSY domains. Resistance, spectral distance and an explicit finite link.

An interaction supported at the origin agrees with $-\partial_x^2$ at every other point. That simple requirement leaves much more freedom than the familiar potential $g\delta(x)$. The freedom lives in the way the two sides of the wave function are joined.

This chapter works with those joining conditions from several directions. Current conservation gives an intuitive calculation, extension theory gives the complete operator classification, and the scattering matrix gives an immediately usable physical coordinate. Regularising potentials and generalised functions then ask how that coordinate is realised. Finally, first-order factors and distances let us put some geometric flesh on the boundary data.

4.1 General contact theory

4.1.1 Derivation from probability conservation

It is convenient to fold the line into two leads, both parametrised by distance $r > 0$ from the origin. Order the right lead first and put

$$\Phi = \begin{pmatrix} \psi(0+) \\ \psi(0-) \end{pmatrix}, \quad \Phi' = \begin{pmatrix} \psi'(0+) \\ -\psi'(0-) \end{pmatrix}. \quad (4.A1)$$

The prime in Φ' points away from the defect on both leads. The boundary form of the second derivative is

$$\langle \psi, H^* \chi \rangle - \langle H^* \psi, \chi \rangle = \Phi_\psi^\dagger \Phi'_\chi - (\Phi'_\psi)^\dagger \Phi_\chi. \quad (4.3)$$

The diagonal version is the difference of currents at the contact. A self-adjoint domain makes the form vanish for every pair of states and is maximal with that property.

On a finite boundary chart the condition can be written

$$\Phi' = K\Phi, \quad K = K^\dagger. \quad (4.1)$$

There are four real parameters. Carreau's notation, with α referring to the left and β to the right, gives

$$K = \begin{pmatrix} \rho + \beta & -\rho e^{-i\vartheta} \\ -\rho e^{i\vartheta} & \rho + \alpha \end{pmatrix}, \quad \rho \geq 0, \quad \alpha, \beta \in \mathbb{R}. \quad (4.4)$$

The phase ϑ completes the four parameters. Direct substitution into the current makes the conjugate off-diagonal terms cancel. Conversely, polarisation of the vanishing boundary form gives Hermiticity [50, 51].

The associated closed quadratic form is

$$q[\psi] = \int_{\mathbb{R} \setminus \{0\}} |\psi'|^2 dx + \Phi^\dagger K \Phi \quad (4.A2)$$

on $H^1(\mathbb{R}_+) \oplus H^1(\mathbb{R}_-)$. The trace inequality bounds its finite-dimensional boundary term relative to the bulk kinetic energy and the norm, so every finite K gives a lower-bounded Hamiltonian.

The chart includes complete separation when $\rho = 0$. The two Robin conditions then act independently. At $\vartheta = 0$, increasing ρ penalises the jump of the wave function through $\rho|\psi(0+) - \psi(0-)|^2$. In the limit $\rho \rightarrow \infty$, continuity is enforced and the remaining coefficient gives the ordinary delta with $g = \alpha + \beta$.

On a finite interval the same boundary-form calculation permits couplings between its two endpoints. Whether that represents a box, a ring, or a device with an external link is a question about the physical model. At the puncture of a line the two traces already sit at the same coordinate position. The boundary relation specifies their coupling; a microscopic geometry can carry further information about how it is achieved.

4.1.2 Derivation by self-adjoint extension

Restrict the free expression to $C_c^\infty(\mathbb{R} \setminus \{0\})$ and take its closure, $H_{\min}$. It is the direct sum of two minimal half-line operators, each with deficiency indices $(1, 1)$. The total indices are therefore $(2, 2)$ [52, 53, 54].

For illustration, the deficiency solutions on each folded lead are proportional to

$$e^{-e^{-i\pi/4}r} \in \ker(H_{\min}^* - i), \quad (4.8)$$

$$e^{-e^{i\pi/4}r} \in \ker(H_{\min}^* + i). \quad (4.9)$$

Both decay at infinity. A unitary map between the two two-dimensional deficiency spaces selects a self-adjoint extension.

The boundary coordinates give a particularly useful global parametrisation. For a fixed reference length $\ell > 0$,

$$\boxed{(U - \mathbb{I})\Phi + i\ell(U + \mathbb{I})\Phi' = 0, \quad U \in U(2).} \quad (4.A3)$$

Equivalently, write $A\Phi + B\Phi' = 0$ with

$$AB^\dagger = BA^\dagger, \quad \text{rank}(A \ B) = 2. \quad (4.A4)$$

The finite Hermitian chart is

$$K = \frac{i}{\ell}(U + \mathbb{I})^{-1}(U - \mathbb{I}). \quad (4.A5)$$

If U has an eigenvalue -1 , the corresponding channel is Dirichlet and is represented by its projector. In full generality,

$$P_D \Phi = 0, \quad Q\Phi' = K_Q Q\Phi, \quad Q = \mathbb{I} - P_D, \quad K_Q = K_Q^\dagger \text{ on } \text{ran } Q. \quad (4.A6)$$

The reference length changes the coordinate U for a fixed physical boundary relation. It supplies no fifth physical parameter.

(2026) This global organisation makes the geometry of boundary conditions explicit and extends naturally to junctions and more general boundaries [55, 56, 57]. A choice of domain is a dynamical parameter of the model, on the same footing as a coefficient in a regular potential.

4.1.3 An integral formulation

The delta jump can be found by integrating the equation once. More generally, let $p_\varepsilon(x) > 0$ be a kinetic coefficient and define the flux $j_\varepsilon = p_\varepsilon \psi'_\varepsilon$. The stationary equation

$$-(p_\varepsilon \psi'_\varepsilon)' + V_\varepsilon \psi_\varepsilon = E \psi_\varepsilon \quad (4.20)$$

becomes the first-order integral system

$$j_\varepsilon(x) = j_\varepsilon(x_0) + \int_{x_0}^x (V_\varepsilon - E) \psi_\varepsilon dy, \quad \psi_\varepsilon(x) = \psi_\varepsilon(x_0) + \int_{x_0}^x p_\varepsilon^{-1} j_\varepsilon dy. \quad (4.14)$$

This displays two independent mechanisms: concentrated potential changes the flux, while a concentrated integral of p^{-1} changes the wave-function value at approximately fixed flux.

For a shrinking interval with vanishing interior probability mass and $V_\varepsilon = 0$, suppose

$$\int_{-\varepsilon}^{\varepsilon} p_\varepsilon(x)^{-1} dx \longrightarrow B > 0. \quad (4.A7)$$

Then j becomes continuous and the wave function acquires the jump Bj . This is the derivative-continuous contact. The concrete choice $p_\varepsilon = 2\varepsilon/B$ in the interval gives the quadratic-form construction described by [equation \(4.A16\)](#).

The old formal expression $p = (1 + B\delta)^{-1}$ is a shorthand for such a family. The integral system gives a direct meaning to the relevant limit through ordinary positive functions. It also shows exactly which quantity survives the squeezing.

4.1.4 The scattering matrix of a boundary relation

On the folded leads write

$$\psi(r) = a_{\text{in}} e^{-ikr} + a_{\text{out}} e^{ikr}. \quad (4.A8)$$

Then $\Phi = a_{\text{in}} + a_{\text{out}}$ and $\Phi' = ik(a_{\text{out}} - a_{\text{in}})$. Substitution into [equation \(4.1\)](#) gives the port-ordered matrix

$$\boxed{\mathcal{S}(k) = (ik\mathbb{I} - K)^{-1}(ik\mathbb{I} + K)}. \quad (4.A9)$$

It is unitary for real k , since K is Hermitian. In a Robin eigenchannel with eigenvalue m ,

$$s_m(k) = \frac{ik + m}{ik - m} = \frac{k - im}{k + im}. \quad (4.A10)$$

Neumann has $s = 1$, Dirichlet has $s = -1$, and $m < 0$ gives a bound state at $E = -m^2$.

The dictionary to Chapter 1 is

$$\mathcal{S} = Sq. \quad (4.A11)$$

In particular, free transmission is $S = \mathbb{I}$ and $\mathcal{S} = q$. Separated Neumann and Dirichlet are $\mathcal{S} = +\mathbb{I}$ and $-\mathbb{I}$, respectively.

For the Carreau chart put

$$D = \alpha\beta + \rho(\alpha + \beta) - k^2 - ik(\alpha + \beta + 2\rho). \quad (4.A12)$$

The four amplitudes are

$$\begin{aligned} t_L &= -\frac{2ik\rho e^{-i\vartheta}}{D}, & t_R &= -\frac{2ik\rho e^{i\vartheta}}{D}, \\ r_R &= \frac{-k^2 - \alpha\beta - \rho(\alpha + \beta) + ik(\beta - \alpha)}{D}, & r_L &= \frac{-k^2 - \alpha\beta - \rho(\alpha + \beta) + ik(\alpha - \beta)}{D}. \end{aligned} \quad (4.25)$$

The compact matrix formula is usually the better calculation tool; these rational expressions make its relation to the original parameters explicit.

4.1.5 Subfamilies worth keeping at hand

Every connected contact can be represented as

$$\begin{pmatrix} \psi(0+) \\ \psi'(0+) \end{pmatrix} = e^{i\varphi} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \psi(0-) \\ \psi'(0-) \end{pmatrix}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1. \quad (4.A13)$$

The pairs (φ, M) and $(\varphi + \pi, -M)$ describe the same relation. The dimensions are $[b] = L$, $[c] = L^{-1}$; a, d, φ are dimensionless.

For incidence from the left,

$$t_L(k) = \frac{2ike^{i\varphi}}{bk^2 + ik(a + d) - c}, \quad r_L(k) = \frac{bk^2 + ik(d - a) + c}{bk^2 + ik(a + d) - c}. \quad (4.A14)$$

The bound-state equation is

$$b\kappa^2 + (a + d)\kappa + c = 0, \quad \kappa > 0. \quad (4.A15)$$

Separated conditions are represented directly by [equation \(4.A6\)](#) or diagonal K , with independent reflection amplitudes on the two leads.

The ordinary delta. For $M = D(g) = \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix}$, continuity and the jump $[\psi'] = g\psi(0)$ give [equation \(1.A9\)](#). The transmission probability is $4k^2/(g^2 + 4k^2)$.

The derivative-continuous contact. For

$$M = J(B) = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}, \quad (4.35)$$

the derivative is continuous and $[\psi] = B\psi'$. In the notation of the Spanish text this was $\delta^{(0)}$; here it is δ'_{int} . Its amplitudes are

$$t = \frac{2}{2 - ikB}, \quad r = \frac{-ikB}{2 - ikB}, \quad \mathcal{T} = \frac{4}{4 + B^2k^2}. \quad (4.40)$$

For $B < 0$ there is one bound state at $-4/B^2$. For $B > 0$ its form is

$$q_B[\psi] = \int_{\mathbb{R} \setminus \{0\}} |\psi'|^2 dx + \frac{1}{B} |\psi(0+) - \psi(0-)|^2. \quad (4.A16)$$

The even channel is Neumann and the odd one is Robin. Its complementary high- and low-energy behaviour with the ordinary delta will be particularly clear in the RG plot.

The scale-invariant discontinuity. For

$$M_\theta = \text{diag}(\theta, \theta^{-1}), \quad \theta \neq 0, \quad (4.43)$$

one obtains

$$t = \frac{2\theta}{1 + \theta^2}, \quad r_L = \frac{1 - \theta^2}{1 + \theta^2}, \quad r_R = -r_L. \quad (4.42)$$

The amplitudes are constant in momentum. This is the real scale-invariant family associated with the conventional Kurasov delta-prime prescription. The notation through boundary conditions identifies it unambiguously.

A Chernoff–Hughes family. A useful lower-triangular family is

$$\begin{pmatrix} \psi(0+) \\ \psi'(0+) \end{pmatrix} = \begin{pmatrix} e^{-z} & 0 \\ r(e^{\bar{z}} - e^{-z}) & e^{\bar{z}} \end{pmatrix} \begin{pmatrix} \psi(0-) \\ \psi'(0-) \end{pmatrix}, \quad z \in \mathbb{C}, \quad r \in \mathbb{R}. \quad (4.32)$$

Writing $z = u + iv$ extracts the common phase e^{-iv} and leaves a real determinant-one matrix. The formal delta-squared descriptions used for this family are tied to a regularisation prescription [58]. The matching relation itself is already a complete operator definition.

4.2 The regularisation problem

There are two distinct convergence questions. A potential can converge as a distribution, and its Hamiltonian can converge as an operator. The first tests V_ε against fixed smooth functions. The second also follows the changing wave functions through the shrinking region.

We use norm-resolvent convergence when

$$\|(H_\varepsilon - z)^{-1} - (H - z)^{-1}\| \rightarrow 0, \quad z \in \mathbb{C} \setminus \mathbb{R}. \quad (4.A17)$$

Strong-resolvent convergence asks for this limit on each fixed vector. The norm condition is stronger. A common lower spectral bound is an additional condition, treated in the next chapter.

4.2.1 Dipoles and the zero-mean scaling

For the antisymmetric pair

$$V_\varepsilon = \lambda \varepsilon^{-\alpha} \{\delta(x - \varepsilon) - \delta(x + \varepsilon)\}, \quad (4.44)$$

the leading zero-energy transfer product is

$$D(v) \begin{pmatrix} 1 & 2\varepsilon \\ 0 & 1 \end{pmatrix} D(-v) = \begin{pmatrix} 1 - 2\varepsilon v & 2\varepsilon \\ -2\varepsilon v^2 & 1 + 2\varepsilon v \end{pmatrix}, \quad v = \lambda \varepsilon^{-\alpha}. \quad (4.51)$$

This immediately identifies the critical exponent $\alpha = 1/2$. Below it the limit is free; at it the delta coefficient is $-2\lambda^2$; above it the generic limiting scattering separates into Dirichlet half-lines [59]. The lower-left entry has remembered the product of the two large strengths even though their distributional sum has zero mean.

For a smooth zero-mean profile W , consider

$$H_\varepsilon = -\partial_x^2 + \varepsilon^{-\alpha} W(x/\varepsilon), \quad \int W = 0. \quad (4.47)$$

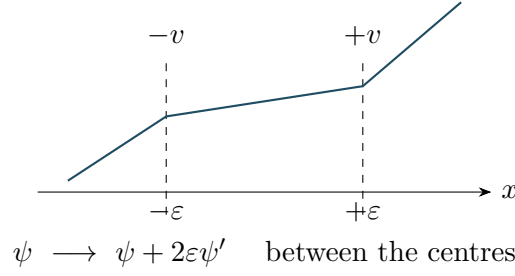


Figure 4.1: The leading linear propagation between two approaching delta centres, with the jump signs of [equation \(4.51\)](#). For a positive wave value and $v > 0$, the negative left delta decreases the slope and the positive right delta increases it. The transfer product retains the quadratic strength term.

In the weak inner-potential regime $\alpha < 2$, the next transfer term has order $\varepsilon^{3-2\alpha}$. At $\alpha = 3/2$ it gives a finite delta coefficient

$$A = \frac{1}{2} \iint W(x)|x-y|W(y)dx dy = - \int \left(\int_{-\infty}^x W(y)dy \right)^2 dx. \quad (4.48)$$

Thus $\alpha < 3/2$ gives the free limit, $\alpha = 3/2$ the indicated attractive delta, and the generic $3/2 < \alpha < 2$ limit is separated Dirichlet.

(2026) At the critical inner scaling $\alpha = 2$, the full cell operator $-\partial_y^2 + W(y)$ enters. A bounded zero-energy solution gives a transmitting diagonal contact [equation \(4.43\)](#); absence of that resonance gives generic decoupling [60, 61]. Stronger scalings can pass through successive resonant regimes and require the actual cell transfer matrix. The separate 2026 article gives the full resonance calculation and its detuning.

This is a useful lesson in what a scaling argument measures. The integral and first moments control a weak inner problem. At order-one inner strength, a zero-energy solution of the whole profile becomes the decisive datum.

4.2.2 Carreau's construction

Carreau constructs a short-range differential interaction with piecewise first-derivative terms [50]. In his convention the kinetic term is $-\frac{1}{2}\partial_x^2$; multiplying the full operator by two puts it into our energy units. The coefficients are controlled by a function $K^\varepsilon(\eta)$ satisfying

$$K^\varepsilon e^{-2\varepsilon K^\varepsilon} = \eta, \quad (4.54)$$

on the branch that diverges as $\varepsilon \rightarrow 0$. For real positive η and small ε ,

$$K^\varepsilon(\eta) = -\frac{1}{2\varepsilon} W_{-1}(-2\varepsilon\eta) = \frac{1}{2\varepsilon} [-\log(2\varepsilon\eta) + \log(-\log(2\varepsilon\eta)) + \dots]. \quad (4.55)$$

The sign in the exponential is fixed by the diverging branch. Complex parameters are obtained by continued branch choices.

His finite-width stationary matching tends to the four-parameter boundary relation. The approximating first-derivative operator is generally non-Hermitian at finite width, so this is a construction in that broader differential-operator class. It illustrates explicitly how a boundary relation can emerge from a shrinking first-order term. For microscopic self-adjoint scalar approximants, the later multi-layer and resonant constructions provide the appropriate route [62, 63, 64].

The special derivative-continuous contact becomes especially simple in the Carreau construction: the interaction consists of opposite first-derivative coefficients on the two halves of the shrinking interval. This is the differential version of allowing a jump at fixed flux in the integral system above.

4.2.3 Regularisation on other spaces

A punctured line already has two boundary ports. One may also place a short connector, a stub, or a junction in the shrinking region. The comparison dynamics on the distant leads stays free while the interior geometry supplies additional boundary parameters.



Figure 4.2: Two embeddings with the topology of a line, redrawn from original Figure 4.2. The topology records their connectivity. A metric and a differential operator specify the additional geometric data.

The corner and the smooth bend in figure 4.2 are homeomorphic. The angle of the drawing by itself supplies no vertex Hamiltonian: the operator, its domain and its metric measure fix that dynamics. The separate finite segment in panel (a) of figure 4.3 carries another piece of information. Its internal spectrum belongs to a compact component, while the infinite line supplies the two scattering channels. Attaching that segment to the line would produce a stub and a different connectivity.

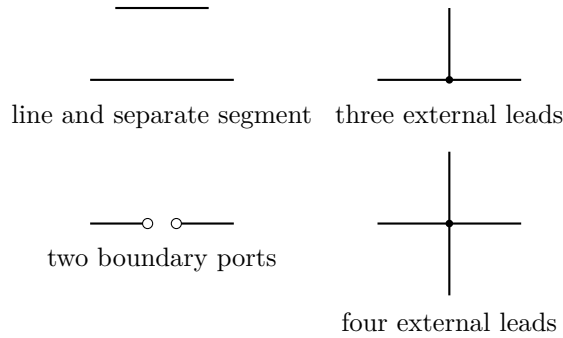


Figure 4.3: Several one-dimensional geometries with different internal connectivity, following original Figure 4.3. Panel (a) has an infinite line and a disconnected finite segment; the other panels have three external leads, two separated half-lines, and four external leads. The external scattering matrix counts asymptotic channels, while compact components have their own spectral data.

(2026) Quantum graphs provide the intrinsic version of this language [65, 57]. A general n -lead self-adjoint vertex has a $U(n)$ boundary description. Shrinking internal edges with tuned delta couplings and localised gauge fields can approximate general vertex couplings in norm resolvent [66]. Thin Neumann waveguides give related geometric contact limits through identification maps between their varying Hilbert spaces [67].

The algebra of functions and its representation specify which geometric features survive the reduction. **(2026)** The inverse question on a star graph also has a direct scattering

answer: its self-adjoint boundary conditions can be reconstructed from the scattering matrix at a prescribed energy [99]. A two-port external matrix can describe several different interiors. Their finite-energy contact limits can coincide while their internal spectra and finite-range corrections differ.

4.3 Potentials containing generalised functions

4.3.1 Integrated jump equations

Let $[f] = f(0+) - f(0-)$ and $\{f\} = (f(0+) + f(0-))/2$. For an ordinary distribution-valued second derivative,

$$D^2\psi = \psi''_{\text{reg}} + [\psi']\delta + [\psi]\delta'. \quad (4.A18)$$

If a concentrated interaction $\rho\psi$ has a specified meaning, integrating once and twice gives the formal matching identities

$$[\psi'] = \langle \rho, \psi \rangle, \quad (4.63)$$

$$[\psi] = -\langle \rho, x\psi \rangle, \quad (4.64)$$

with the bounded-energy interior mass terms vanishing under the required estimates. Griffiths' calculations motivate this approach for delta derivatives [68, 142, 107, 125].

For discontinuous ψ , the product prescription is an essential part of these identities. A regularising sequence supplies one prescription through its own changing wave functions. A distributional boundary model can instead state a trace convention directly. These are two concrete ways of defining the calculation.

4.3.2 Splitting the test space

The natural test space on the punctured line has independent smooth one-sided traces:

$$\mathcal{D}_* = C_c^\infty((-\infty, 0]) \oplus C_c^\infty([0, \infty)). \quad (4.71)$$

Its topology controls derivatives of every order on fixed compact sets on both sides. Besides the average delta functional, it admits the jump functional

$$\delta_*(\varphi) = \{\varphi\}, \quad \mathcal{J}(\varphi) = [\varphi]. \quad (4.76)$$

For example, differentiating the integration functional on this split space gives $D1 = \mathcal{J}$. Integration by parts for a piecewise smooth ψ gives

$$\langle D\psi, \varphi \rangle = \int \psi'_{\text{reg}} \varphi dx + [\psi]\{\varphi\} + \{\psi\}[\varphi]. \quad (4.75)$$

On continuous ordinary test functions the last term vanishes. On the doubled space it records precisely the extra boundary information.

The conventional symmetric boundary prescription used below is

$$\psi\delta = \{\psi\}\delta, \quad \psi\delta' = \{\psi\}\delta' - \{\psi'\}\delta. \quad (4.A19)$$

These equations specify the trace product in the distributional differential expression. If one retains the full split-space dual and defines multiplication by direct duality, additional jump-functional terms appear. For example $\delta_*(\psi\varphi) = \{\psi\}\{\varphi\} + [\psi][\varphi]/4$. Stating the convention keeps the boundary algebra explicit [69].

4.3.3 Kurasov's operator description

Kurasov's four-real-parameter differential expression can be written

$$L_X = -D^2(1 + X_4\delta) + iD(2X_3\delta - iX_4\delta') + X_1\delta + (X_2 - iX_3)\delta'. \quad (4.77)$$

The distributional construction and its maximal L^2 domain give the self-adjoint matching relations [69]. Put

$$\Delta_X = (2 - iX_3)^2 + X_1X_4 - X_2^2. \quad (4.A20)$$

For $\Delta_X \neq 0$ the matching matrix is

$$\begin{pmatrix} \psi(0+) \\ \psi'(0+) \end{pmatrix} = \frac{1}{\Delta_X} \begin{pmatrix} (2 + X_2)^2 - X_1X_4 + X_3^2 & -4X_4 \\ 4X_1 & (2 - X_2)^2 - X_1X_4 + X_3^2 \end{pmatrix} \begin{pmatrix} \psi(0-) \\ \psi'(0-) \end{pmatrix}. \quad (4.78)$$

The real numerator has determinant $|\Delta_X|^2$. Factoring $|\Delta_X|/\Delta_X$ therefore gives the phase times real determinant-one form of [equation \(4.A13\)](#).

At $\Delta_X = 0$, the relations are separated. For $X_4 \neq 0$, they read

$$\psi'(0+) = \frac{X_2 - 2}{X_4}\psi(0+), \quad \psi'(0-) = \frac{X_2 + 2}{X_4}\psi(0-). \quad (4.79)$$

For $X_4 = 0, X_2 = 2$, the conditions are $\psi(0-) = 0$ and $\psi'(0+) = X_1\psi(0+)/4$. For $X_4 = 0, X_2 = -2$, they are $\psi(0+) = 0$ and $\psi'(0-) = -X_1\psi(0-)/4$.

These formulas make several special roles visible. X_1 gives the ordinary delta. X_2 gives the diagonal scale-invariant discontinuity. A pure X_3 gives a phase. A pure X_4 gives the derivative-continuous matrix $J(-X_4)$. The distributional formula contains the kinetic and first-derivative terms needed to represent all these boundary relations in one symbolic language.

4.3.4 The generalised Griffiths equations in practice

For the conventional expression $-D^2 + A\delta + B\delta'$ with [equation \(4.A19\)](#), cancelling the singular terms gives

$$[\psi] = B\{\psi\}, \quad (4.85)$$

$$[\psi'] = A\{\psi\} - B\{\psi'\}. \quad (4.84)$$

For $B \neq \pm 2$ these are equivalent to

$$\begin{pmatrix} \psi(0+) \\ \psi'(0+) \end{pmatrix} = \begin{pmatrix} (2 + B)/(2 - B) & 0 \\ 4A/(4 - B^2) & (2 - B)/(2 + B) \end{pmatrix} \begin{pmatrix} \psi(0-) \\ \psi'(0-) \end{pmatrix}. \quad (4.83)$$

At $B = \pm 2$ the un-divided equations [equation \(4.85\)](#) give the separated endpoint conditions. This is a useful place to keep the original jump equations instead of pushing a singular parameter through a matrix denominator.

The derivative-continuous contact uses a different boundary term: its jump is proportional to the common derivative. It has the quadratic form [equation \(4.A16\)](#) and the variable-kinetic realisation [equation \(4.A7\)](#). A rank-one nonlocal approximation with a renormalised derivative-delta form factor supplies another realisation [59]. The different formulas describe different microscopic mechanisms for the same specified boundary relation.

4.4 Supersymmetry of point interactions

The delta example showed how a first-order product domain creates a contact. Let us keep the factor general:

$$A = Z(x)\partial_x + W(x), \quad (4.91)$$

$$A^\dagger = -\partial_x Z(x)^* + W(x)^*. \quad (4.92)$$

Its formal product is

$$A^\dagger A = -\partial_x |Z|^2 \partial_x + (W^* Z - Z^* W) \partial_x + |W|^2 - (Z^* W)'. \quad (4.93)$$

Let $Z = e^{i\theta_\pm}$ and $W = \mu_\pm e^{i\theta_\pm}$ be constant on each half-line, with real $\mu_\pm$. If $|\mu_+| = |\mu_-| = \mu$, the bulk product is $-\partial_x^2 + \mu^2$. The interaction is entirely in the domain.

Choose

$$\mathcal{D}(A) = \{\psi \in H^1(\mathbb{R}_+) \oplus H^1(\mathbb{R}_-) : \psi(0+) = \lambda \psi(0-)\}, \quad \lambda \neq 0. \quad (4.99)$$

Integration by parts gives the adjoint condition

$$\chi(0+) = \frac{e^{i(\theta_+ - \theta_-)}}{\bar{\lambda}} \chi(0-). \quad (4.100)$$

Impose it on $\chi = A\psi$. The phase of Z cancels in the product relation, leaving

$$\psi(0+) = \lambda \psi(0-), \quad (4.104)$$

$$\psi'(0+) = \frac{\mu_- - |\lambda|^2 \mu_+}{\bar{\lambda}} \psi(0-) + \frac{1}{\bar{\lambda}} \psi'(0-). \quad (4.103)$$

For real λ this is a lower-triangular delta/delta-prime contact; a complex λ includes its common phase. Taking $\mu_+ = -\mu_-$ gives a coefficient proportional to $1 + |\lambda|^2$, while equal $\mu_\pm$ gives $1 - |\lambda|^2$. The two choices recover the two useful branches of the original construction.

Other first-order domains give separated factors. Leaving both traces free makes the adjoint traces vanish; the product imposes $\psi'(0\pm) + \mu_\pm \psi(0\pm) = 0$. A zero trace on one side and a free trace on the other gives a Dirichlet/Robin pair. Independent half-line energy shifts accommodate arbitrary separated Robin parameters. A single global SUSY Hamiltonian uses the compatible common shift.

The simple Z of unit modulus produces the lower-triangular family directly. Allowing a variable positive $|Z|^2$ makes the kinetic coefficient participate, and its shrinking weak-link limit produces $J(B)$. The separate 2026 article gives a stable constant-kinetic scalar construction using a positive zero-energy profile.

(2026) Point-singularity SUSY requires the supercharge domain itself to satisfy the grading and adjoint conditions. Uchino and Tsutsui analyse this issue for singular boundary domains [70]. The first-order factors above display explicitly where those conditions enter, before any partner spectrum is compared.

4.5 Metric aspects

The geometric chapter associated a radial operator with a known metric. Here the direction of thought is reversed: begin with an operator and ask which distance its commutator with functions measures. Connes' distance formula is

$$d_D(x, y) = \sup\{|f(x) - f(y)| : f = f^*, \|[D, f]\| \leq 1\}. \quad (4.117)$$

The algebra, its representation and the self-adjoint operator D are the data of the construction [71, 72].

4.5.A1 The line and its phase contact (2026)

For $D = -i\partial_x$ on the ordinary line, $[D, f] = -if'$, so the norm constraint is the usual Lipschitz bound. The supremum gives $d(x, y) = |x - y|$.

Now use the self-adjoint momentum extension with $\psi(0+) = e^{i\varphi}\psi(0-)$. Multiplication by f preserves this domain when $f(0+) = f(0-)$. Its admissible functions therefore retain a common value at the contact, and the same derivative bound gives the ordinary line distance. Adding a multiplication superpotential to a first-order expression leaves this commutator unchanged when the algebra and compatible domain are held fixed.

On two completely independent leads, the algebra can instead contain independent constants. Their difference is unconstrained by a purely differential commutator, and the distance between the components is infinite. The separation is encoded in the algebra and domain together.

4.5.A2 A finite link with an explicit spectral operator (2026)

(2026-gpt) A finite distance between two boundary points has a particularly simple complete model. Represent $\mathbb{C} \oplus \mathbb{C}$ diagonally on $\mathbb{C}^2$ and take

$$D_F = \begin{pmatrix} 0 & m \\ \bar{m} & 0 \end{pmatrix}, \quad m \neq 0. \quad (4.A21)$$

For $f = \text{diag}(f_+, f_-)$,

$$\|[D_F, f]\| = |m| |f_+ - f_-|, \quad d_F(+, -) = |m|^{-1}. \quad (4.A22)$$

Every ingredient is specified and the supremum is attained. A continuum product construction with a constant finite link gives the corresponding two-sheet geometry [73]. A metric graph with two rays and a connector of length ℓ gives the geodesic distance $r + s + \ell$ between points at distances r, s from the connector. In both cases a link operator or an actual edge supplies the finite scale.

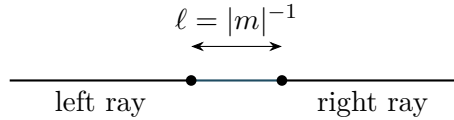


Figure 4.4: An explicit finite-link geometry. **(2026-gpt)** The length is supplied by a specified finite spectral operator or a connector edge. The contact resistance is a separate datum unless a microscopic model relates the two.

4.5.A3 The weak-link resistance and the operator metric (2026)

(2026-gpt) The geometric content of that picture can be calculated explicitly. For a free interval of length $L > 0$,

$$F_L(k) = \begin{pmatrix} \cos kL & \sin(kL)/k \\ -k \sin kL & \cos kL \end{pmatrix}, \quad F_L(0) = J(L). \quad (4.A23)$$

Thus the derivative-continuous contact retains the linear, zero-energy relation illustrated in figure 4.5. A connector of finite length retains its full energy-dependent matrix. The

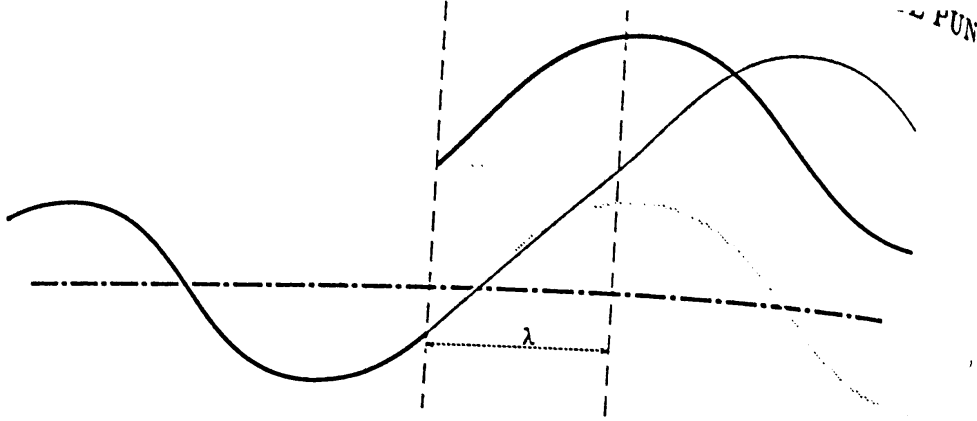


Figure 4.5: The original linear-extrapolation picture (Figure 4.4), restored in monochrome. The straight continuation crosses an interval marked λ and suggests $[\psi] = \lambda\psi'$. The exact zero-energy propagation matrix is $J(\lambda)$. A finite-energy connector carries the additional propagation terms in [equation \(4.A23\)](#).

probability stored inside the connector is part of its state space. This gives a precise bridge from the original gap metaphor to the internal-defect actions in the separate 2026 article.

(2026-gpt) For a positive kinetic coefficient, a first-order factor has principal part $\sqrt{p} \partial_x$. Its commutator norm bounds $\sqrt{p} |f'|$, so the associated one-dimensional intrinsic length is

$$L_D = \int p^{-1/2} dx. \quad (4.A24)$$

The zero-energy resistance in [equation \(4.A7\)](#) is instead

$$B = \int p^{-1} dx. \quad (4.A25)$$

For the shrinking layer $p = 2\varepsilon/B$ on $[-\varepsilon, \varepsilon]$,

$$\int_{-\varepsilon}^{\varepsilon} p^{-1} dx = B, \quad \int_{-\varepsilon}^{\varepsilon} p^{-1/2} dx = \sqrt{2\varepsilon B} \rightarrow 0. \quad (4.A26)$$

The resistance survives while the intrinsic metric length collapses. This calculation completes the comparison suggested by the original finite-gap picture: the two operational lengths are determined by different powers of the microscopic coefficient.

(2026) The positive form [equation \(4.A16\)](#) has a natural probabilistic counterpart in snapping-out Brownian motion and semipermeable interfaces [74]. The interface jump is proportional to flux, and the parameter controls permeability. This supplies a concrete interpretation of $B > 0$ directly from the boundary form.

4.5.A4 Circles, phases and holonomy (2026)

(2026) Closing the line into a circle makes the global role of a contact phase especially clear. Fülöp and Tsutsui classify a $U(2)$ family for a particle on a circle and show that the spectral parameters can occupy a smaller space than the operator parameters [98]. The identification of an operator with its energy levels alone therefore has visible limits even in this elementary geometry.

Martinetti calculates spectral distance for covariant Dirac operators on bundles over the circle [100]. The holonomy of the connection determines the connected components of that distance. This supplies a precise later version of the thesis’s interest in parallel transport at a puncture: a global connection and its representation enter the metric calculation. On an isolated line a relative phase can be gauged away; a closed route records its holonomy.

4.A1 Physical realisations developed since 1995 (2026)

Transverse confinement gives effective one-dimensional contact couplings from a higher-dimensional short-range interaction. Olshanii’s confined-atom calculation is a central example [75]. Its confinement-induced scale and resonance provide a physical route to the effective coupling, with a stated range of validity below transverse excitations.

The even/odd contact distinction also enters the one-dimensional Bose–Fermi duality of Cheon and Shigehara [76]. The change of exchange symmetry is accompanied by a change in contact conditions. In a many-body setting the coincidence hyperplanes and their compatibility carry the extra structure required by the several-particle problem.

The point interaction is therefore a useful common language for several microscopic pictures: concentrated scalar layers, altered kinetic coefficients, thin junctions, confined atoms and explicit boundary links. The next chapter develops the original scattering-space change of scale. The separate 2026 article compares microscopic realisations and their stability conditions.

Chapter 5

The renormalisation group

Earlier work, renormalisation and changes of scale. A Wilson-style construction in scattering space. Fixed points and perturbations. Comparison with the known point interactions. Flows of regularising families and the available microscopic operators.

The absence of an intrinsic range in a point interaction invites us to try scaling. The immediate inspiration comes from renormalisation, already used for contacts in two and three dimensions [83, 84, 85, 87]. Here the regularising Hamiltonians themselves are part of the question. It is convenient to begin with a space of interactions and let the scattering matrix supply its coordinates [1].

We compare finite-range theories with a reference range a_0 . Their explicit Hamiltonians need not appear in every calculation; an inverse problem can recover them when the required scattering data are available. This follows the spirit of the Wilson construction [77], with some of the reference-scale bookkeeping suppressed. Quantum-mechanical path-integral approaches give a complementary starting point [79].

The calculation in this chapter is the thesis's scaling construction: fixed matrices, linearised directions, known trajectories and regularising examples. We keep the transmission-diagonal convention of Chapter 1, $S = \begin{pmatrix} t_L & r_R \\ r_L & t_R \end{pmatrix}$. The expanded treatment, including the matrix beta function, dimensional transmutation and microscopic action classification, is contained in the separate [2026 article](#).

5.1 Renormalisation and change of scale

Start with an interaction carrying a range a and dilate that range. We obtain a line of theories describing the same physics in different units. The useful operation is to take a sequence of decreasing-range interactions and dilate its members back to one reference range. The resulting sequence can then be compared in one space.

The existence of a limit remains a question about the sequence. The flow tells us where a convergent, appropriately tuned sequence can land. Near a fixed point, irrelevant displacements shrink and relevant displacements grow. Repeated dilation can consequently leave a finite displacement along a trajectory departing from that fixed point. The marginal coordinates specify which fixed point supplies the departure.

The picture is a useful way to organise a limiting calculation. Its application to operators also requires a stated convergence topology and control of the shrinking region. The regularisation discussion in Chapter 4 supplies that analytical setting.

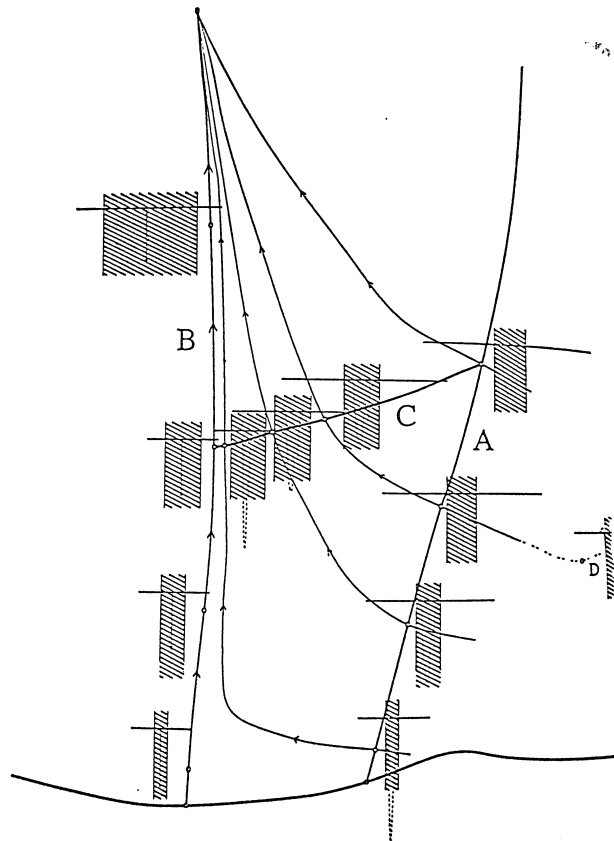


Figure 5.1: Flows in a space of interactions, restored from original Figure 5.1. The bare sequence A is transformed into a renormalised sequence C approaching the trajectory B . The limiting bare point, labelled a in the Spanish caption and D in the drawing, flows towards the fixed point b from which B departs.

5.2 Flow in the space of scattering matrices

5.2.1 Implementation

Assign an admissible scattering function to each interaction. Unitarity and the momentum-reversal relation are those of Section 1.2.2. Analytic continuation and threshold behaviour carry the further information about its range. With t increasing towards the infrared, the dilation is simply

$$(T_t S)(k) = S(e^{-t} k). \quad (5.1)$$

Two points on one such trajectory represent the same interaction viewed at different momentum scales. This is the usual gossip about renormalisation, here written as a substitution in a matrix function.

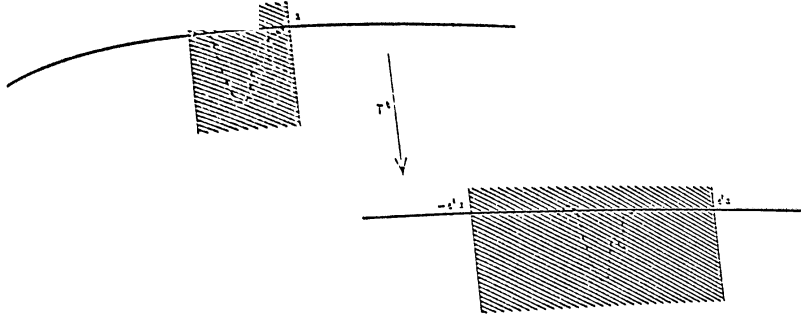


Figure 5.2: The change-of-scale metaphor, restored from original Figure 5.2. The shaded intervals illustrate the moving cutoff. On complete scattering data the dilation has the inverse T_{-t} ; a restriction to an observable window supplies the associated loss of information.

5.2.2 Fixed points

A fixed matrix is independent of k . Combining unitarity with the momentum-reversal relation gives the mixed family and two isolated points,

$$\left\{ I_{\theta, \phi} = \begin{pmatrix} e^{i\theta} \cos \phi & -\sin \phi \\ \sin \phi & e^{-i\theta} \cos \phi \end{pmatrix} \right\} \cup \{+q, -q\}. \quad (5.2)$$

The parametrisation has the familiar coordinate identifications at its poles; the mixed component is a two-sphere. Time reversal imposes equal left and right transmission and selects

$$\left\{ \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \right\} \cup \{+q, -q\}. \quad (5.3)$$

The circle is the scale-invariant discontinuity family of Chapter 4, completed by its separated limits. Its dimensionless parameter carries no length. The isolated points separate the half-lines: $+q$ is Neumann and $-q$ is Dirichlet. Free transmission is the identity on the mixed component.

5.2.3 Relevant, irrelevant and marginal directions

Perturb a fixed matrix while keeping unitarity:

$$S_{\mathbf{a}}(k) = S_0 \exp \left(\sum_{j=0}^3 a_j(k) L_j \right), \quad a_j(k) \in \mathbb{R}. \quad (5.4)$$

To first order,

$$S_{\mathbf{a}} - S_0 = \sum_j a_j(k) S_0 L_j + O(|\mathbf{a}|^2). \quad (5.5)$$

Choose the four anti-Hermitian generators

$$L_0 = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \quad L_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (5.6)$$

For example, at $+q$ the tangent matrices are

$$qL = \left\{ \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right\}. \quad (5.7)$$

At the remaining fixed matrices the rule is just multiplication:

$$(-q)L = -qL, \quad I_{0,0}L = L, \quad I_{\theta,\phi}L = \{I_{\theta,\phi}L_j\}_{j=0}^3. \quad (5.8)$$

The generators supply the unitary tangent space. The momentum and range requirements still have to be imposed on its coefficient functions.

An infinitesimal dilation gives

$$T_{\delta t} S(k) = S(k) + \delta t \mathcal{O}_k S(k) + O(\delta t^2), \quad (5.9)$$

where

$$\mathcal{O}_k S(k) = \partial_t S(e^{-t}k) \Big|_{t=0} = -k \partial_k S(k). \quad (5.10)$$

An eigendirection of the linearised dilation obeys

$$-k \sum_j a'_j(k) S_0 L_j \delta t = (\lambda - 1) \sum_j a_j(k) S_0 L_j. \quad (5.11)$$

Writing the eigenvalue and its coefficient function as

$$\lambda = 1 + n\delta t + O(\delta t^2), \quad (5.12)$$

we obtain

$$\mathbf{a}(k) = k^{-n} \mathbf{a}_0. \quad (5.13)$$

Positive n is relevant in this infrared orientation, negative n irrelevant, and $n = 0$ marginal. The saddle-point picture says precisely this: some displacements approach the fixed point while others depart from it.

At the free point $S_0 = I$, momentum reversal gives

$$a_0(-k) = -a_0(k), \quad a_1(-k) = -a_1(k), \quad a_2(-k) = a_2(k), \quad a_3(-k) = a_3(k). \quad (5.14)$$

This parity statement refers to the chosen tangent basis at I . Keeping the lowest powers gives the six formal directions

$$(0, 0, 1, 0) \quad \text{marginal}, \quad \lambda = 1, \quad (5.15)$$

$$(0, 0, 0, 1) \quad \text{marginal}, \quad \lambda = 1, \quad (5.16)$$

$$(k, 0, 0, 0) \quad \text{irrelevant}, \quad \lambda = 1 - \delta t, \quad (5.17)$$

$$(0, k, 0, 0) \quad \text{irrelevant}, \quad \lambda = 1 - \delta t, \quad (5.18)$$

$$(k^{-1}, 0, 0, 0) \quad \text{relevant}, \quad \lambda = 1 + \delta t, \quad (5.19)$$

$$(0, k^{-1}, 0, 0) \quad \text{relevant}, \quad \lambda = 1 + \delta t. \quad (5.20)$$

For an even short-range interaction, the generic threshold laws used in the original argument are

$$\tan \delta_e(k) = O(k^{-1}), \quad (5.21)$$

$$\tan \delta_o(k) = O(k). \quad (5.22)$$

Threshold resonances require the exceptional branches described in Chapter 1. Moreover, threshold orders alone leave finite-range corrections available. The two discarded displays of the original calculation are retained here with a strike-through:

$$\text{---}(k, k, 0, 0) \text{---} \text{excluded by short range alone} \quad (5.23)$$

$$\text{---}(k^{-1}, -k^{-1}, 0, 0) \text{---} \text{excluded by short range alone} \quad (5.24)$$

The four-dimensional result applies to energy-independent self-adjoint point domains. The boundary conditions of Chapter 4 select that manifold. Around free transmission their tangent directions are

$$(0, 0, 1, 0) \quad \text{marginal}, \quad \lambda = 1, \quad (5.25)$$

$$(0, 0, 0, 1) \quad \text{marginal}, \quad \lambda = 1, \quad (5.26)$$

$$(k, k, 0, 0) \quad \text{irrelevant}, \quad \lambda = 1 - \delta t, \quad (5.27)$$

$$(k^{-1}, -k^{-1}, 0, 0) \quad \text{relevant}, \quad \lambda = 1 + \delta t. \quad (5.28)$$

The first two move along the fixed surface. The third is the direction in which the derivative-continuous contact arrives at free transmission. The last generates the ordinary delta. Let us check these identifications against the actual solutions; a small matrix is usually more persuasive than a large collection of names.

5.3 Flow of the known solution

5.3.1 Constructing the trajectories

The scale transformation sends k to a positive multiple of k . For the point interactions of Chapter 4 the whole trajectory remains in the same family, so its endpoints follow from the limits at zero and infinite momentum.

Use Carreau's parameters, with the right boundary trace first:

$$K = \begin{pmatrix} \rho + \beta & -\rho e^{-i\theta} \\ -\rho e^{i\theta} & \rho + \alpha \end{pmatrix}. \quad (5.29)$$

Put $D(k) = \alpha\beta + \rho(\alpha + \beta) - k^2 - ik(\alpha + \beta + 2\rho)$. The scattering matrix is

$$S(k) = \frac{1}{D(k)} \begin{pmatrix} -2ik\rho e^{-i\theta} & -k^2 - \alpha\beta - \rho(\alpha + \beta) + ik(\beta - \alpha) \\ -k^2 - \alpha\beta - \rho(\alpha + \beta) + ik(\alpha - \beta) & -2ik\rho e^{i\theta} \end{pmatrix}. \quad (5.30)$$

The range coordinate a gives the dimensionless family

$$\tilde{S}_a(p) = S(p/a), \quad p = ak. \quad (5.31)$$

In the following ultraviolet expansions $a \downarrow 0$ at fixed $p \neq 0$; the remainder is $O(a^2)$. Write $R_\theta = \text{diag}(e^{-i\theta}, e^{i\theta}) = I_{-\theta,0}$.

For $\rho = 0$, the two half-lines are independent and the trajectory leaves $+q$ along

$$\tilde{S}_a - q = -\frac{2ia}{p} \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix} + O(a^2). \quad (5.32)$$

For $\alpha = \beta = 0$, it leaves $+q$ along

$$\tilde{S}_a - q = -\frac{2i\rho a}{p} \begin{pmatrix} -e^{-i\theta} & 1 \\ 1 & -e^{i\theta} \end{pmatrix} + O(a^2). \quad (5.33)$$

At $\theta = 0$ this is the derivative-continuous contact. For $\rho \rightarrow \infty$ at fixed $g = \alpha + \beta$, the delta family leaves R_θ along

$$\tilde{S}_a - R_\theta = -\frac{iga}{2p} \begin{pmatrix} e^{-i\theta} & 1 \\ 1 & e^{i\theta} \end{pmatrix} + O(a^2). \quad (5.34)$$

Canonical time reversal sets the relative phase to zero and gives the usual delta.

For infrared arrivals put $k = \bar{a}p$ and let $\bar{a} \downarrow 0$. For $\alpha = \beta = 0$ and $\rho \neq 0$,

$$S(\bar{a}p) - R_\theta = -\frac{i\bar{a}p}{2\rho} \begin{pmatrix} -e^{-i\theta} & 1 \\ 1 & -e^{i\theta} \end{pmatrix} + O(\bar{a}^2). \quad (5.35)$$

For $\rho = 0$ and $\alpha\beta \neq 0$,

$$S(\bar{a}p) + q = -2i\bar{a}p \begin{pmatrix} 0 & \beta^{-1} \\ \alpha^{-1} & 0 \end{pmatrix} + O(\bar{a}^2). \quad (5.36)$$

For the delta family with $g \neq 0$,

$$S(\bar{a}p) + q = -\frac{2i\bar{a}p}{g} \begin{pmatrix} e^{-i\theta} & 1 \\ 1 & e^{i\theta} \end{pmatrix} + O(\bar{a}^2). \quad (5.37)$$

Zero Robin coefficients keep their Neumann channels and use the corresponding mixed endpoint. The coupling constants enter together with the reference scale, as they should: a dimensionful constant has to be measured in some units before it can appear on a dimensionless trajectory.

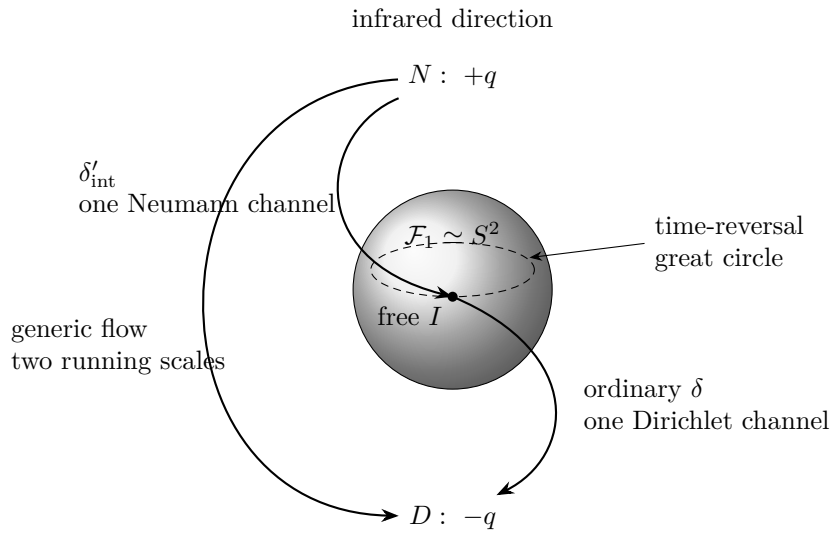


Figure 5.3: Flow in the space of short-range interactions, reconstructed from original Figure 5.3. The mixed fixed surface sits between the Neumann source $+q$ and the Dirichlet sink $-q$. The ordinary delta goes from I to $-q$; the derivative-continuous contact goes from $+q$ to I . The two endpoint labels in the photographed drawing have been corrected to agree with its Spanish caption and the amplitudes. The monochrome archive retains the printed drawing.

5.3.2 Analysis and comparison

The limits of the explicit trajectories agree with the fixed matrices, and their leading displacements agree with the perturbative directions. The marginal directions move from one fixed interaction to a neighbouring fixed interaction: they are exactly marginal. In the time-reversal-invariant problem they trace a circle; including the relative phase completes the surface.

The irrelevant directions in this contact model are arrival directions of trajectories departing from other fixed points. The simple examples make that structure visible without requiring a microscopic potential for every point of the drawing. Changing the reflection signs of a known scattering matrix can lead outside this particular contact sector. The inverse-square tails of the family in Section 1.4.4 illustrate how a changed asymptotic class can carry the additional behaviour.

5.4 Flow of regularising families

The renormalisation triangle separates two operations: changing a bare coupling and dilating the resulting theory. We can now see both operations in a concrete regularisation.

5.4.1 The usual regularisation

Take the odd pair

$$V_a(x) = \frac{g}{2a} \{\delta(x+a) - \delta(x-a)\}. \quad (5.38)$$

For incidence from the left, use $e^{ikx} + r_L e^{-ikx}$ before the pair, $Ae^{ikx} + Be^{-ikx}$ between its centres, and te^{ikx} after it. With $u = g/(2a)$ the matching equations at the first centre are

$$\begin{aligned} Ae^{-ika} + Be^{ika} &= e^{-ika} + r_L e^{ika}, \\ ik\{(A-1)e^{-ika} - (B-r_L)e^{ika}\} &= u(e^{-ika} + r_L e^{ika}). \end{aligned} \quad (5.39)$$

At the second centre they are

$$\begin{aligned} Ae^{ika} + Be^{-ika} &= te^{ika}, \\ ik\{(t-A)e^{ika} + Be^{-ika}\} &= -ute^{ika}. \end{aligned} \quad (5.40)$$

Solving gives

$$\begin{aligned} t_a(k) &= \left[1 - \frac{g^2}{16a^2 k^2} (e^{4ika} - 1) \right]^{-1}, \\ r_{L,a}(k) &= t_a(k) \sin(2ak) \left(-\frac{g}{2ak} + \frac{ig^2}{8a^2 k^2} \right), \\ r_{R,a}(k) &= t_a(k) \sin(2ak) \left(+\frac{g}{2ak} + \frac{ig^2}{8a^2 k^2} \right). \end{aligned} \quad (5.41)$$

Thus $S_a(k) \rightarrow -q$ at fixed $g \neq 0$ and $k \neq 0$ when $a \rightarrow 0$. In reference-range coordinates the same family is

$$\tilde{S}_g(p) = S_a(p/a), \quad \tilde{t}_g(p) = \left[1 - \frac{g^2}{16p^2} (e^{4ip} - 1) \right]^{-1}. \quad (5.42)$$

Its dependence on a has disappeared. Every member is in the basin of the same infrared endpoint:

$$\lim_{p \rightarrow 0} \tilde{S}_g(p) = -q \quad (g \neq 0). \quad (5.43)$$

The naive squeeze has consequently landed at separation. To retain a finite interaction we have to tune the bare sequence as well.

5.4.2 Renormalisation of the coupling constant

Let $g(a) \rightarrow 0$. The reference-range sequence now approaches free transmission. Apply a dilation whose momentum factor is

$$\tau(a) = \alpha g(a)^2, \quad \alpha > 0. \quad (5.44)$$

The limit follows directly from the small- p expansion of the exponential:

$$\lim_{a \rightarrow 0} (T_{-\log \tau(a)} \tilde{S}_{g(a)})(p) = \frac{1}{1 + 4i\alpha p} \begin{pmatrix} 4i\alpha p & -1 \\ -1 & 4i\alpha p \end{pmatrix}. \quad (5.45)$$

This is an attractive delta. In the reference coordinate its strength is $-1/(2\alpha)$; restoring the reference length gives $-1/(2\alpha a_0)$. The sign and the factors of i are fixed by the unitary amplitudes. Choosing the usual momentum factor $\tau(a) = a/a_0$ gives

$$g(a) = \pm \sqrt{\frac{a}{\alpha a_0}}. \quad (5.46)$$

Both signs produce the same limiting attractive coupling. This is the critical dipole scaling discussed in Section 4.2.1 [52].

5.4.3 Regularisation with admissible potentials

For a generic fixed short-range profile the endpoints are

$$S(k \rightarrow 0) = -q, \quad (5.47)$$

$$S(k \rightarrow \infty) = I. \quad (5.48)$$

The dipole example shows how tuning a bare family towards the free fixed point can retain a delta trajectory. These two endpoint values describe the untuned profile; by themselves they do not determine every tuned shrinking-support limit. The exceptional threshold alternative of Section 1.2.3 and the scaling hypotheses in Section 4.2.1 remain part of the argument. With those hypotheses attached, the renormalisation picture gives the geometric reading of the regularisation theorem used here.

5.4.4 Exotic regularisers

The original discussion also points towards altered kinetic terms, velocity-dependent or nonlocal interactions, and potentials outside the weighted short-range classes. Each changes a specific ingredient of the microscopic problem. Intermediate inverse-square tails, already encountered in Chapter 1, can alter the threshold behaviour. The modified kinetic coefficient of Chapter 4 supplies another explicit mechanism.

There is also an instructive delta approximation built from the Hilbert transform. With the normalisation displayed explicitly, take

$$V_\varepsilon(x) = \frac{1}{\pi^2 x} \log \left| \frac{\varepsilon + x}{\varepsilon - x} \right|. \quad (5.49)$$

The value at zero is understood by continuity. Its logarithmic singularities at $\pm\varepsilon$ are locally integrable, and the function has tails beyond that interval. Define

$$(\mathbf{H}\psi)(x) = \frac{1}{\pi} \text{pv} \int_{\mathbb{R}} \frac{\psi(y)}{x-y} dy, \quad F = i\mathbf{H}, \quad F^2 = I. \quad (5.50)$$

Composing F with the version of F whose kernel omits $|x-y| < \varepsilon$ gives the convolution kernel (5.49). This fixes the sign and normalisation in the original suggestion.

For $|x| < \varepsilon$, an antiderivative can be written with the dilogarithm:

$$\int V_\varepsilon(x) dx = \frac{\text{Li}_2(x/\varepsilon) - \text{Li}_2(-x/\varepsilon)}{\pi^2} + C. \quad (5.51)$$

Its elementary series is

$$\int_0^x V_\varepsilon(s) ds = \frac{2}{\pi^2} \sum_{n=0}^{\infty} \frac{(x/\varepsilon)^{2n+1}}{(2n+1)^2}. \quad (5.52)$$

Hence

$$\int_{-\varepsilon}^{\varepsilon} V_\varepsilon(x) dx = \frac{1}{2}, \quad \int_{\mathbb{R}} V_\varepsilon(x) dx = 1, \quad V_\varepsilon \longrightarrow \delta \quad \text{in distributions.} \quad (5.53)$$

The outside tails carry the other half of the mass. The shrinking, positive approximate identity makes the distributional limit transparent. A Hamiltonian approximation still calls for its own operator argument, as in Chapter 4.

Finally, Šeba's nonlocal construction uses the formal expression

$$H = -\partial_x^2 + [\lambda|\delta'\rangle\langle\delta'|]_{\text{ren}}. \quad (5.54)$$

A regularised form factor and a running coefficient define the bracketed term [59]. Its distributional sign can be kept straight through

$$|\delta'\rangle\langle\delta'| = -D\delta D, \quad V = -\lambda D\delta D \quad \text{formally.} \quad (5.55)$$

Indeed $\langle\delta', \psi\rangle = -\psi'(0)$. This formal identity connects the rank-one language with the derivative-coefficient language; their regularised couplings and domains supply the actual operator. The chapter ends with precisely that collection of microscopic choices around the same point-interaction problem.

Conclusions

Potential scattering has given us a large landscape to move through. The scattering matrix has supplied the common thread, from elementary line potentials to geometric motion and contact renormalisation.

The first useful lesson is the value of a formulation. The matrix Jost description carries familiar radial-scattering tools onto the full line. Parity, connection coefficients and analytic continuation make the examples easy to compare. The inverse equations specify the reconstruction data, including the discrete norming information.

The second is the reach of factorisation. For a rank-one symmetric space, the radial measure supplies the superpotential. For the delta and transparent ladders, the first-order intertwiner controls bound levels and scattering amplitudes at once. The domain follows the calculation and keeps the geometry visible.

The numerical work distinguishes mechanisms that look similar in a plot. A strongly coupled regular well localises around its minimum; the Coulomb singularity selects its own dilation. Resonance pairs meet at square-root branch points. The double delta has an exact Lambert description, and the triple delta produces different coalescence heights. The formulas give the pictures their units and interpretation.

The contact problem has been especially instructive. Current conservation, extension theory, the boundary form and the scattering matrix all describe the same four-parameter operator family. Generalised functions give a symbolic language for its domains. Positive kinetic layers, resonant scalar profiles and several approaching centres give concrete microscopic realisations. A finite-link spectral operator supplies an explicit metric construction, and the weak-link calculation separates its intrinsic length from its zero-energy resistance.

Finally, renormalisation organises the known contacts in scattering space. Its fixed matrices include the scale-invariant surface and the two separated contacts. Relevant, irrelevant and marginal displacements identify the departures, arrivals and motions along that surface. The explicit delta and derivative-continuous trajectories check the perturbative construction. The odd dipole illustrates how a tuned bare coupling and a dilation combine to retain a finite interaction.

For me, much of the pleasure of this work lies in seeing the same small calculations turn up in several places. A Wronskian becomes a transmission coefficient; a radial density becomes a superpotential; a shrinking interval becomes a boundary form. Algebra and geometry meet most usefully when the factorisation is written down and its domain is carried with it. That is the point at which the several threads of the thesis come together.

Computational procedures

The original calculations

The bound-state equation was integrated by shooting. Initial conditions at one endpoint were adjusted until the solution had the required behaviour at the other. For a symmetric potential, parity reduces the task to a half-line: start with a Dirichlet or Neumann condition at zero and match to a decaying exponential outside the interaction region.

The 1995 calculations used modified Fortran programs by Salcedo and collaborators [135]. The related numerical-method paper by Oset and Salcedo is recorded separately [136]. The programs originally returned eigenvalues and eigenfunctions; the versions used here concentrated on the eigenvalues. Coupling dependence was inserted into the potential routine, and a VAX DCL batch procedure recompiled and ran the calculation through a sequence of coupling values. The phase shifts were obtained through a related integration and asymptotic matching procedure.

The output was naturally ordered by coupling. A small post-processing program reordered it by level, producing the trajectories displayed in the plots. It also wrote the input format for PROLOT and launched the graphics program. This division of labour was perfectly serviceable: Fortran integrated, the batch system kept the sequence moving, and a separate plotting program worried about the picture.

The pole maps were calculated in Mathematica from the implicit equations $J(k, g) = 0$. In the simplest cases one could solve for a parameter and plot directly. In the others the implicit complex equation supplied the root search. A small imaginary part of the coupling fixed a continuation convention when trajectories crossed a branch boundary [137, 141].

Mathematica also provided cross-checks of determinants, matrix inversions, boundary-coordinate changes and long algebraic expressions. Numerical substitution was useful beside symbolic simplification. A sign that survives two descriptions of the same scattering matrix is a little more trustworthy than one that survives only a long printed line.

The thesis was composed with L^AT_EX 2.09 and L^AT_EX 2_ε under SunOS and Linux. The PostScript output was printed on a LaserWriter Personal NT. The change of machines did little to alter the author's relationship with typographical errors.

Additional calculations (2026)

(2026) The figures in this edition are regenerated from the stated equations. The source package contains a Python script using NumPy, SciPy and Matplotlib, a separate reproduction of the RG calculations, the numerical verification records and vector PDF figures.

For the regular full-line wells, a three-point finite-difference Hamiltonian is diagonalised on $[-16, 16]$ with 2399 interior points. The spacing is $1/75$. The five lowest eigenvalues are followed over the displayed coupling interval. The exact sech^2 ground state provides a

calibration at several couplings. Levels within $\kappa = 0.08$ of threshold are suppressed in the figure, since finite-box effects are largest there.

For Yukawa, the radial Dirichlet problem is discretised on $(0, 30)$ with spacing 0.0025. The first three levels are calculated. The Hulthén and Coulomb curves use their exact formulas, which also check the coupling and energy normalisations. These plots illustrate fixed-index coupling behaviour; threshold precision depends on the chosen spatial box and is treated separately from the pole root calculations.

The Pöschl–Teller phases use the gamma-function amplitudes in [equation \(3.29\)](#). Phases are unwrapped from high momentum. The numerical identity $|t|^2 + |r|^2 = 1$ is checked at the plotted coupling values. This also checks the relative normalisation of the reflection and transmission formulas.

The double-delta poles use the exact Lambert branches. The roots introduced by multiplying an equation by k are removed by checking the original Jost function. The square and triple-delta maps use independent complex root searches over a grid of real couplings, followed by residual and symmetry checks. The double-zero values solve both $J = 0$ and $\partial_k J = 0$. A figure of root locations is distinguished from the local square-root expansion, which is determined analytically by [equation \(3.A14\)](#).

The reflectionless examples use the determinant formula [equation \(1.A13\)](#); the derivatives are evaluated through matrix trace identities. If $M = \mathbb{I} + A(x)$,

$$\partial_x^2 \log \det M = \text{Tr}(M^{-1} A'' - M^{-1} A' M^{-1} A'). \quad (1)$$

The independent norming parameters generate the two spatial profiles. Their common transmission is calculated directly from their common bound-state momenta.

The supplied verification records cover the numerical figures and the analytical formulas used in their reconstruction. The separate article supplies its own RG and connector calculations. Additional consistency checks of the restored Chapter 5 verify the Carreau amplitudes, all six trajectory expansions, the odd-dipole matching equations and its critically tuned delta limit. The source-numbering record accounts for all 343 original equations.

All figure calculations use $\hbar^2/(2m) = 1$. Widths and centre positions are stated in the captions or equations. The graphical data are generated from those same conventions, keeping the analytical and numerical parts of the calculation on the same scale.

Notation

Symbol	Meaning
$H_0 = -\partial_x^2$	Free Hamiltonian on the stated domain; units $\hbar^2/(2m) = 1$.
$E = k^2$	Complex energy parametrised by momentum k . A bound state has $k = i\kappa$, $E = -\kappa^2$.
S	Transmission-diagonal full-line scattering matrix in Chapters 1–3 and 5: $\begin{pmatrix} t_L & r_R \\ r_L & t_R \end{pmatrix}$.
$q = \sigma_x$	Matrix exchanging the two channels.
$\mathcal{S} = Sq$	Folded boundary-port scattering matrix, with right port first.
e, o	Even and odd parity channels, with amplitudes $t + r$ and $t - r$ in the transmission-diagonal convention.
f_+, f_-	Jost solutions outgoing at the right and left ends.
J_e, J_o	Parity Jost functions in the specified regular endpoint normalisation.
Φ, Φ'	Boundary values and derivatives pointing away from the contact, as in equation (4.A1) .
K	Hermitian Robin boundary matrix; P_D separately specifies the Dirichlet channels.
U	Global $U(2)$ boundary coordinate, with reference length ℓ .
M	Real determinant-one transfer matrix in a connected contact, with a separately written overall phase.
$D(g), J(B)$	Lower- and upper-triangular elementary contact matrices.
δ'_{int}	Derivative-continuous discontinuity contact, $[\psi] = B\psi'$. The Spanish thesis used $\delta^{(0)}$.
M_θ	Scale-invariant diagonal contact, $\text{diag}(\theta, \theta^{-1})$.
$[f], \{f\}$	Jump and average of the one-sided traces at zero.
A	First-order SUSY factor. The products and adjoints include their domains.
p, q in Chapter 2	Root multiplicities $m_\alpha, m_{2\alpha}$; the second is a scalar parameter, distinct from the channel-exchange matrix.
$\rho = p/2 + q$	Half-sum of positive roots with multiplicities in the rank-one radial formulas.
W_n	Branch n of the Lambert function, $W_n e^{W_n} = z$.
L_σ^1	Weighted integrable class $\int (1 + x ^\sigma) V(x) dx < \infty$.
(2026)	Substantive later development integrated into the English edition.
(2026-gpt)	Additional calculation or construction developed in this edition.

Appendix A

Figures from the Spanish edition

The photographed figure pages preserve the original drawings, including their arrows, root-multiplicity map and explanatory sketches. The companion [monochrome figure archive](#) contains all nineteen numbered figures and the additional boundary-link sketch, with English captions. The source package contains the processed images and the recorded crops. The original photographic pagination differs from the later Spanish PDF pagination; the figure numbers provide the common reference.

Original	Subject	Location in this edition
2.1	Successive SUSY pairs	Figure 2.1
2.2	Full multiplicity-plane map	Figure 2.2 and equation (2.A13)
3.1	Four regular wells	Figure 3.1
3.2	Yukawa and Hulthén	Figure 3.2
3.3	Coulomb	Figure 3.2 and equation (3.A3)
3.4	Odd and even phases	Figure 3.3
3.5	Even square-well poles	Figure 3.4 and equation (3.A16)
3.6	Even square-barrier poles	Figure 3.4
3.7	Single-delta pole	Figure 3.5
3.8	Even double-delta poles	Figure 3.6 and equation (3.A19)
3.9	Odd double-delta poles	Figure 3.6 and equation (3.A19)
3.10	Even triple-delta poles	Figure 3.7
4.1	Linear propagation through a dipole	Figure 4.1 and equation (4.51)
4.2	Corner and smooth bend	Figure 4.2
4.3	Components and junctions	Figure 4.3
4.4	Gap and linear extrapolation	Figure 4.5 and equation (4.A23)
5.1	Bare and renormalised paths	Figure 5.1
5.2	Change of scale and cutoff	Figure 5.2 and equation (5.1)
5.3	Global fixed surface	Figure 5.3
Link sketch	Two rays and a distance parameter	Figure 4.4

Chapter 5 keeps the transmission-diagonal convention of the Spanish thesis. Its reconstructed global diagram corrects the two isolated endpoint labels to agree with the original caption. The separate article gives the expanded channel and action analysis.

Appendix B

Source equation concordance

(2026) Original equation numbers belong to the Spanish edition and remain reserved throughout this translation. Added equations carry an A, for example (1.A1); added sections and subsections use the same signal. Corrections retain the original equation number. The original section hierarchy of Chapters 1–5 is preserved.

A separate source display that has been removed from the running derivation is reproduced below with a strike-through. This records an editorial withdrawal of that display; its mathematical content may have been consolidated into another formula or into the surrounding explanation. A strike-through by itself makes no claim that the original identity was false. Equations (5.23)–(5.24) carry their withdrawal directly in Chapter 5. This concordance accounts for all 343 numbered source equations.

Number	Status in this edition
(1.1)	Retained at (1.1), page 4.
(1.2)	Retained at (1.2), page 4.
(1.3)	Retained at (1.3), page 4.
(1.4)	Retained at (1.4), page 4.
(1.5)	Retained at (1.5), page 4.
(1.6)	Retained at (1.6), page 5.
(1.7)	Retained at (1.7), page 5.
(1.8)	Retained at (1.8), page 5.
(1.9)	Retained at (1.9), page 5.
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(1.13)	Retained at (1.13), page 5.
(1.14)	Retained at (1.14), page 5.
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(1.16)	Retained at (1.16), page 6.
(1.17)	Retained at (1.17), page 6.
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(1.92)	Retained at (1.92), page 13.
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~~(4.24)~~	Separate display withdrawn; facsimile on page 79.
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Withdrawn source displays

Source (1.10) Spanish PDF page 34; separate display withdrawn.

$$\cancel{G_k^J(x, y) = \frac{1}{k} \sin k(x - y)} \quad (1.10)$$

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$$\cancel{\varphi_k(x) = \chi_k(x) F_k} \quad (1.18)$$

Source (1.19) Spanish PDF page 35; separate display withdrawn.

$$\cancel{\psi_k = \begin{pmatrix} T_k^l & 0 \\ 0 & T_k^r \end{pmatrix} \chi_k} \quad (1.19)$$

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$$\cancel{\psi_k^+(x) = \psi_k^-(x) S_k} \quad (1.20)$$

Source (1.21) Spanish PDF page 36; separate display withdrawn.

$$\cancel{\varphi_k = \varphi_{-k} q} \quad (1.21)$$

Source (1.22) Spanish PDF page 36; separate display withdrawn.

$$\cancel{\psi_k = \psi_{-k} q S_k} \quad (1.22)$$

Source (1.23) Spanish PDF page 36; separate display withdrawn.

$$\cancel{\psi_{-k} = \varphi_{-k} J_{-k}^{-1} = \varphi_k q J_{-k}^{-1} =} \quad (1.23)$$

Source (1.24) Spanish PDF page 36; separate display withdrawn.

$$\cancel{= \psi_k J_k q J_{-k}^{-1}} \quad (1.24)$$

Source (1.29) Spanish PDF page 37; separate display withdrawn.

$$\begin{array}{l} T^r \leftrightarrow T^l \\ \overline{R^r \leftrightarrow R^r} \\ R^l \leftrightarrow R^l \end{array} \quad (1.29)$$

Source (1.30) Spanish PDF page 37; separate display withdrawn.

$$\overline{V(x)^\dagger = V(x) \Rightarrow V(x) = V^*(x)} \quad (1.30)$$

Source (1.32) Spanish PDF page 38; separate display withdrawn.

$$\overline{\int dx(1+x^2)|V(x)|dx < +\infty} \quad (1.32)$$

Source (1.36) Spanish PDF page 38; separate display withdrawn.

$$\overline{T(k) \sim k^{\sigma+\rho}} \quad (1.36)$$

Source (1.37) Spanish PDF page 39; separate display withdrawn.

$$\overline{S_k^{asint} = \frac{g^*(k)}{g(k)} S_k^{func}} \quad (1.37)$$

Source (1.38) Spanish PDF page 39; separate display withdrawn.

$$\overline{J_k = \frac{1}{T_k} F_k} \quad (1.38)$$

Source (1.39) Spanish PDF page 39; separate display withdrawn.

$$\overline{J_k = \frac{1}{2ik} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} Wr(\chi_k^T, \varphi_k)} \quad (1.39)$$

Source (1.41) Spanish PDF page 40; separate display withdrawn.

$$\overline{\xi_k \equiv (f_k(x), f_k(-x)), f_k \equiv f_1} \quad (1.41)$$

Source (1.43) Spanish PDF page 40; separate display withdrawn.

$$\overline{\{e^{ikt} \mid k \in \mathbb{R}\}} \quad (1.43)$$

Source (1.44) Spanish PDF page 41; separate display withdrawn.

$$\overline{(i \sin kx, \cos kx),} \quad (1.44)$$

Source (1.45) Spanish PDF page 41; separate display withdrawn.

$$\overline{g_{impar}(0) = 0 \quad g'_{impar}(0) = ik} \quad (1.45)$$

Source (1.46) Spanish PDF page 41; separate display withdrawn.

$$\cancel{g_{\overline{par}}(0) = 1 \qquad g'_{\overline{par}}(0) = 0} \qquad (1.46)$$

Source (1.47) Spanish PDF page 41; separate display withdrawn.

$$\cancel{\xi = (f(x) - f(-x), f(x) + f(-x))} \qquad (1.47)$$

Source (1.48) Spanish PDF page 41; separate display withdrawn.

$$\cancel{J_k = \frac{P}{2ik} W\{\chi, \varphi\} \Big|_{x \downarrow 0} = \frac{1}{2} \left(f_k(x) + f_k(-x) - \frac{1}{ik} (f'_k(-x) + f'_k(x)) \right) \Big|_{x \downarrow 0}} \qquad (1.48)$$

Source (1.52) Spanish PDF page 42; separate display withdrawn.

$$\cancel{f(x) = \begin{cases} x > 0, & e^{ikx} \\ x < 0, & e^{ikx} - \frac{g}{k} \sin kx f(0) \end{cases}} \qquad (1.52)$$

Source (1.59) Spanish PDF page 43; separate display withdrawn.

$$\cancel{V(x) = -ge^{-|x|}} \qquad (1.59)$$

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$$\cancel{f_k(x) = \sum_{n=0}^{\infty} f_k^{(n)}(x); f_k^{(0)}(x) = e^{-ik|x|}} \qquad (1.60)$$

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$$\cancel{f_k^{(n+1)}(x) = \frac{1}{k} \int_x^{\infty} \sin k(y-x) V(y) f_k^{(n)}(y) dy} \qquad (1.61)$$

Source (1.62) Spanish PDF page 43; separate display withdrawn.

$$\cancel{f_k(x) = e^{-ikx} \sum_{n=0}^{\infty} C_k^{(n)} e^{-nx},} \qquad (1.62)$$

Source (1.63) Spanish PDF page 43; separate display withdrawn.

$$\cancel{C_k^{(n)} = \frac{-g}{n(2ik+n)} C_k^{(n-1)}} \qquad (1.63)$$

Source (1.67) Spanish PDF page 44; separate display withdrawn.

$$\cancel{e^{2i(\delta_+(k)+kb)} \equiv \frac{1+i \tan(\delta_+ + kb)}{1-i \tan(\delta_+ + kb)} \equiv \frac{1-ih_+(k)/k}{1+ih_+(k)/k},} \qquad (1.67)$$

Source (1.69) Spanish PDF page 44; separate display withdrawn.

$$J_{-k}^+ = e^{+ikb} \cos(kb) \left(1 + i \frac{h_+(k)}{k}\right), \quad (1.69)$$

Source (1.71) Spanish PDF page 44; separate display withdrawn.

$$J_k^- = e^{+ikb} \sin kb \left(h_-(k)/k + i\right) \quad (1.71)$$

Source (1.76) Spanish PDF page 45; separate display withdrawn.

$$1/T_k = o(|k|^{\sigma-2}) \quad (1.76)$$

Source (1.78) Spanish PDF page 46; separate display withdrawn.

$$L(\alpha) = K(\alpha) + \int_0^x d\beta K(\alpha + \beta) Q L(\beta) Q \quad (1.78)$$

Source (1.79) Spanish PDF page 46; separate display withdrawn.

$$L(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk q(S_k - I) J_k q e^{ik\alpha} \quad (1.79)$$

Source (1.80) Spanish PDF page 46; separate display withdrawn.

$$K(\alpha) = \frac{1}{2\pi} q \int_{-\infty}^{\infty} dk q(S_k - I) e^{ik\alpha} q \quad (1.80)$$

Source (1.82) Spanish PDF page 47; separate display withdrawn.

$$\frac{d(\rho_E - \rho_E^0)}{dE} = \begin{cases} (1/4\pi k)(J_k^\dagger J_k)^{-1} - I] & \text{para } E > 0 \\ \sum_b M_b \delta(E - E_b) & \text{para } E < 0 \end{cases} \quad (1.82)$$

Source (1.85) Spanish PDF page 47; separate display withdrawn.

$$J_k = I^n J_k^{red} \prod_n (\Gamma_k^{(n)} \hat{I}) \quad (1.85)$$

Source (1.86) Spanish PDF page 47; separate display withdrawn.

$$\Gamma_k = 1 - B + B \frac{k + iK}{k - iK}, \text{ con } B = \tilde{r} r \hat{I} / r \hat{I} \tilde{r} \quad (1.86)$$

Source (1.90) Spanish PDF page 48; separate display withdrawn.

$$(A^{-1} \frac{d}{dx} A \frac{d}{dx} + k^2) \Psi(x) = 0 \quad (1.90)$$

Source (1.93) Spanish PDF page 49; separate display withdrawn.

$$\cancel{V(x,k) = 2\delta(x + \frac{\pi}{2k}).} \quad (1.93)$$

Source (2.4) Spanish PDF page 55; separate display withdrawn.

$$\cancel{a = (\frac{m_\alpha + m_{2\alpha} - 1}{2})^2 = (\frac{m_{2\alpha} - 1}{2})^2, b = (m_{2\alpha} - 1)^2 = 1} \quad (2.4)$$

Source (2.5) Spanish PDF page 56; separate display withdrawn.

$$\cancel{\Delta(S^n) = \frac{d^2}{d\theta^2} + (n-1)\cot\frac{d}{d\theta} + \Omega} \quad (2.5)$$

Source (2.7) Spanish PDF page 56; separate display withdrawn.

$$\cancel{\frac{d^2}{dr^2} + \frac{n-1}{r}\frac{d}{dr} + \frac{1}{r^2}\Omega} \quad (2.7)$$

Source (2.9) Spanish PDF page 57; separate display withdrawn.

$$\cancel{\delta_n(k) = \frac{(3-n)\pi}{4}} \quad (2.9)$$

Source (2.11) Spanish PDF page 57; separate display withdrawn.

$$\cancel{Q(r) = \sigma_2 p_r + \sigma_1 W(r) \quad C = \sigma_3} \quad (2.11)$$

Source (2.14) Spanish PDF page 57; separate display withdrawn.

$$\cancel{H_+ = AA^+ \quad H_- = A^+A} \quad (2.14)$$

Source (2.19) Spanish PDF page 59; separate display withdrawn.

$$\cancel{D(A) = D(A^+) = \{\Psi(x) \in AC(\mathbb{R} - \{0\}) | \Psi(0^+) = \Psi(0^-)\}} \quad (2.19)$$

Source (2.25) Spanish PDF page 60; separate display withdrawn.

$$\cancel{A^+|\Psi_n\rangle = |\Phi_{n+1}\rangle} \quad (2.25)$$

Source (2.26) Spanish PDF page 61; separate display withdrawn.

$$\cancel{W^2(r) \pm W'(r) = const.} \quad (2.26)$$

Source (2.35) Spanish PDF page 64; separate display withdrawn.

$$\cancel{A = -\frac{d}{d\theta} + W(\theta) \quad A\Psi_0(\theta) = 0} \quad (2.35)$$

Source (2.37) Spanish PDF page 64; separate display withdrawn.

$$\cancel{W_j \rightarrow W_{j+1} \rightarrow \dots \rightarrow W_k} \quad (2.37)$$

Source (2.40) Spanish PDF page 65; separate display withdrawn.

$$\cancel{\Psi_1^\lambda = A_\lambda^+ \Psi_0^{\lambda-1} = \left(\frac{d}{d\theta} + \lambda \cot \theta \right) \sin^{\lambda-1} \theta} \quad (2.40)$$

Source (3.7) Spanish PDF page 75; separate display withdrawn.

$$\cancel{H_0 = -\frac{d^2}{dr^2} + \frac{l(l+1)}{r^2} - \frac{2g}{r}} \quad (3.7)$$

Source (3.11) Spanish PDF page 78; separate display withdrawn.

$$\cancel{\Psi = A \cos k'x \quad (|x| < b),} \quad (3.11)$$

Source (3.12) Spanish PDF page 78; separate display withdrawn.

$$\cancel{\Psi = B e \cos(kx + \delta_+) \quad (|x| > b),} \quad (3.12)$$

Source (3.13) Spanish PDF page 78; separate display withdrawn.

$$\cancel{S_+(k) = e^{2i\delta_+(k)} = \frac{J^+(-k)}{J^+(+k)}} \quad (3.13)$$

Source (3.16) Spanish PDF page 78; separate display withdrawn.

$$\cancel{k = 0; k = (2n+1)(\pi/2) - i\infty.} \quad (3.16)$$

Source (3.17) Spanish PDF page 79; separate display withdrawn.

$$\cancel{V(x) = g\delta(x),} \quad (3.17)$$

Source (3.18) Spanish PDF page 79; separate display withdrawn.

$$\cancel{f(x) = e^{ikx} \frac{1}{k} \sin(kx) f(0) H(-x),} \quad (3.18)$$

Source (3.19) Spanish PDF page 79; separate display withdrawn.

$$\cancel{f_k(0) = 1 \Rightarrow J^-(k) = 1} \quad (3.19)$$

Source (3.20) Spanish PDF page 80; separate display withdrawn.

$$\cancel{J^+(k) = 1 + \frac{g}{2ik},} \quad (3.20)$$

Source (3.21) Spanish PDF page 81; separate display withdrawn.

$$V(x) = g[\delta(x+a) + \delta(x-a)], \quad (3.21)$$

Source (3.25) Spanish PDF page 83; separate display withdrawn.

$$J^-(k) \equiv 1 + (g/k)e^{ika} \sin ka, \quad (3.25)$$

Source (3.27) Spanish PDF page 84; separate display withdrawn.

$$J(k) \propto \prod_1^l (k-n), \quad (3.27)$$

Source (4.2) Spanish PDF page 89; separate display withdrawn.

$$j(x) = \frac{1}{2i}(\Psi^* \frac{d\Psi}{dx} - \Psi \frac{d\Psi^*}{dx}) \quad (4.2)$$

Source (4.5) Spanish PDF page 90; separate display withdrawn.

$$H_0 \equiv - \frac{d^2}{dx^2} \Big|_{C_0^\infty(\mathbb{R}-\{0\})} \quad (4.5)$$

Source (4.6) Spanish PDF page 90; separate display withdrawn.

$$H_0 = H_{(-\infty,0)}^- \oplus H_{(0,+\infty)}^+ \quad (4.6)$$

Source (4.7) Spanish PDF page 90; separate display withdrawn.

$$D(H_U) = \{f \in L^2(\mathbb{R}), f = \varphi + c_1(\varphi_1^+ + u_{11}\varphi_1^- + u_{12}\varphi_2^-) + c_2(\varphi_2^+ + u_{21}\varphi_1^- + u_{22}\varphi_2^-)\} \quad (4.7)$$

Source (4.10) Spanish PDF page 91; separate display withdrawn.

$$\begin{pmatrix} f'(0_+) \\ f'(0_-) \end{pmatrix} = N \begin{pmatrix} f(0_+) \\ f(0_-) \end{pmatrix} \quad (4.10)$$

Source (4.11) Spanish PDF page 91; separate display withdrawn.

$$N = \frac{a}{1 + \text{tr}U + \det U} \begin{pmatrix} -(1 + u_{22} + iu_{11} + i \det U) & (1 - i)u_{21} \\ -(1 - i)u_{12} & 1 + u_{11} + iu_{22} + i \det U \end{pmatrix} \quad (4.11)$$

Source (4.12) Spanish PDF page 91; separate display withdrawn.

$$N = \begin{pmatrix} \rho + \beta & -\rho e^{-i\theta} \\ \rho e^{i\theta} & -\rho - \alpha \end{pmatrix} \quad (4.12)$$

Source (4.13) Spanish PDF page 91; separate display withdrawn.

$$\Psi'(x) = \int^x (\lambda\delta(z) - E)\Psi(z)dz \quad (4.13)$$

Source (4.15) Spanish PDF page 91; separate display withdrawn.

$$\Psi(x) = \int^x dy(1 + \lambda\delta(y)) \int^y dz(-E)\Psi(z) \quad (4.15)$$

Source (4.16) Spanish PDF page 92; separate display withdrawn.

$$\Psi'(x) = (1 + \lambda\delta(x)) \int^x dz(-E)\Psi(z) \quad (4.16)$$

Source (4.17) Spanish PDF page 92; separate display withdrawn.

$$\frac{d^2}{dx^2}\Psi(x) = \lambda\delta'(x) \int^x dz(-E)\Psi(z) + (1 + \lambda\delta(x))(-E)\Psi(x) \quad (4.17)$$

Source (4.18) Spanish PDF page 92; separate display withdrawn.

$$= \frac{\lambda\delta'(x)}{1 + \lambda\delta(x)} \Psi'(x) + (1 + \lambda\delta(x))(-E)\Psi(x) \quad (4.18)$$

Source (4.19) Spanish PDF page 92; separate display withdrawn.

$$\left(\frac{1}{1 + \lambda\delta(x)} \frac{d^2}{dx^2} - \frac{\lambda\delta'(x)}{(1 + \lambda\delta(x))^2} \frac{d}{dx} \right) \Psi(x) = -E\Psi(x) \quad (4.19)$$

Source (4.21) Spanish PDF page 92; separate display withdrawn.

$$u_1(x) = e^{ikxT^l} \quad x > 0 \quad (4.21)$$

Source (4.22) Spanish PDF page 92; separate display withdrawn.

$$e^{ikx} + e^{-ikx}R^l \quad x < 0 \quad (4.22)$$

Source (4.23) Spanish PDF page 92; separate display withdrawn.

$$u_2(x) = e^{-ikx} + e^{ikx}R^r \quad x > 0 \quad (4.23)$$

Source (4.24) Spanish PDF page 92; separate display withdrawn.

$$e^{-ikxT^r} \quad x < 0 \quad (4.24)$$

Source (4.26) Spanish PDF page 93; separate display withdrawn.

$$\rho = 0, \quad \alpha = \frac{1 + u_{11} + i(u_{22} + u_{11}u_{22})}{1 + u_{22} + (u_{11} + u_{11}u_{22})} \quad (4.26)$$

Source (4.27) Spanish PDF page 93; separate display withdrawn.

$$\beta = -\frac{1 + u_{22} + i(u_{11} + u_{11}u_{22})}{1 + u_{22} + (u_{11} + u_{11}u_{22})} \quad (4.27)$$

Source (4.28) Spanish PDF page 93; separate display withdrawn.

$$S_k = \begin{pmatrix} 0 & \frac{\alpha\beta + i\alpha k - i\beta k + k^2}{\alpha\beta - i\alpha k - i\beta k - k^2} \\ \frac{\alpha\beta - i\alpha k + i\beta k + k^2}{\alpha\beta - i\alpha k - i\beta k - k^2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{\beta + ik}{\beta - ik} \\ \frac{\alpha + ik}{\alpha - ik} & 0 \end{pmatrix} \quad (4.28)$$

Source (4.29) Spanish PDF page 94; separate display withdrawn.

$$\begin{pmatrix} \Psi(0^+) \\ \Psi'(0^+) \end{pmatrix} = \begin{pmatrix} C & D \\ B & A \end{pmatrix} \begin{pmatrix} \Psi(0^-) \\ \Psi'(0^-) \end{pmatrix} \quad (4.29)$$

Source (4.30) Spanish PDF page 94; separate display withdrawn.

$$-B\delta + (1+C)(\delta' - \delta \frac{d}{dx}) - \delta \delta' \frac{d}{dx} \quad (4.30)$$

Source (4.31) Spanish PDF page 94; separate display withdrawn.

$$\theta = 0, \rho = \frac{1}{D}, \beta = \alpha = \frac{A+1}{D} \quad (4.31)$$

Source (4.33) Spanish PDF page 94; separate display withdrawn.

$$\langle \delta' | H(x) \rangle = \int \delta' H = - \int \delta \delta = - \langle \delta^2 | 1 \rangle \quad (4.33)$$

Source (4.34) Spanish PDF page 95; separate display withdrawn.

$$-\Psi(0^+) = \Psi(0^-) = 0 \quad -\Psi'(0^+) = \Psi'(0^-) = A\Psi(0) \quad \text{Dirac-}\delta \quad (4.34)$$

Source (4.36) Spanish PDF page 95; separate display withdrawn.

$$D(A^+A) = \{\Psi \in D(A) | A\Psi \in D(A^+)\} \quad (4.36)$$

Source (4.37) Spanish PDF page 95; separate display withdrawn.

$$D(A^+A) = \{\Psi | \Psi(0) = \Psi(1) = 0\} \quad (4.37)$$

Source (4.38) Spanish PDF page 95; separate display withdrawn.

$$D(AA^+) = \{\Psi | \Psi'(0) = \Psi'(1) = 0\} \quad (4.38)$$

Source (4.39) Spanish PDF page 95; separate display withdrawn.

$$S_k^\delta = S_k(\rho \rightarrow \infty, \theta = 0) = \begin{pmatrix} \frac{2ik}{\alpha + \beta - 2ik} & \frac{\alpha + \beta}{\alpha + \beta - 2ik} \\ \frac{\alpha + \beta}{\alpha + \beta - 2ik} & \frac{2ik}{\alpha + \beta - 2ik} \end{pmatrix} \quad (4.39)$$

Source (4.41) Spanish PDF page 95; separate display withdrawn.

$$\alpha = \mu \sin \phi, \beta = \mu \cos \phi, \rho = \mu \frac{-\sin \phi \cos \phi}{\sin \phi + \cos \phi} \quad (4.41)$$

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$$\begin{aligned} S.R. \lim_{\epsilon \rightarrow 0} H_\epsilon &= H_0 \text{ para } \alpha < 1/2 \\ S.R. \lim_{\epsilon \rightarrow 0} H_\epsilon &= H_A \text{ con } A = -2\lambda^2, \text{ para } \alpha = 1/2 \\ S.R. \lim_{\epsilon \rightarrow 0} H_\epsilon &= H_D \text{ para } \alpha > 1/2 \end{aligned} \quad (4.45)$$

Source (4.46) Spanish PDF page 97; separate display withdrawn.

$$\int_R V(x) dx = 0 \quad (4.46)$$

Source (4.49) Spanish PDF page 98; separate display withdrawn.

$$\psi(-\frac{\epsilon}{2}^+) = \psi_-, \psi'(-\frac{\epsilon}{2}^+) = \psi'_- + \frac{c}{\epsilon^\alpha} \psi_- \quad (4.49)$$

Source (4.50) Spanish PDF page 98; separate display withdrawn.

$$\psi(\frac{\epsilon}{2}^-) = \psi_- + \epsilon(\psi'_- + \frac{c}{\epsilon^\alpha} \psi_-), \psi'(\frac{\epsilon}{2}^-) = \psi'_- + \frac{c}{\epsilon^\alpha} \psi_- \quad (4.50)$$

Source (4.52) Spanish PDF page 99; separate display withdrawn.

$$H^\epsilon = -\frac{1}{2} \frac{d^2}{dx^2} + I^\epsilon \quad (4.52)$$

Source (4.53) Spanish PDF page 99; separate display withdrawn.

$$I^\epsilon = \begin{cases} K^\epsilon(\rho e^{i\theta})(\frac{d}{dx} + \alpha + \rho - \rho e^{i\theta}) & -\epsilon < x < 0 \\ -K^\epsilon(\rho e^{-i\theta})(\frac{d}{dx} + \beta + \rho - \rho e^{-i\theta}) & 0 < x < \epsilon \\ 0 & \text{en cualquier otro punto} \end{cases} \quad (4.53)$$

Source (4.56) Spanish PDF page 99; separate display withdrawn.

$$\left(\frac{-\Psi'(-\epsilon)}{\Psi'(+\epsilon)} \right) = M \left(\frac{\Psi(-\epsilon)}{\Psi(+\epsilon)} \right) \quad (4.56)$$

Source (4.57) Spanish PDF page 99; separate display withdrawn.

$$S_k^\epsilon = e^{-2i\epsilon k} S_k, \quad (4.57)$$

Source (4.58) Spanish PDF page 99; separate display withdrawn.

$$I^\epsilon = \begin{cases} K^\epsilon(\rho) \frac{d}{dx} & -\epsilon < x < 0 \\ -K^\epsilon(\rho) \frac{d}{dx} & 0 < x < \epsilon \\ 0 & \text{en cualquier otro punto} \end{cases} \quad (4.58)$$

Source (4.59) Spanish PDF page 101; separate display withdrawn.

$$-\Psi'_\lambda(x) + \Psi'_\lambda(-\epsilon) + \int_{-\epsilon}^x V_\lambda(y) \Psi_\lambda(y) dy = E_\lambda \int_{-\epsilon}^x \Psi_\lambda(y) dy \quad (4.59)$$

Source (4.60) Spanish PDF page 101; separate display withdrawn.

$$x \int_{-\epsilon}^x V_\lambda(z) \Psi_\lambda(z) dz - \int_{-\epsilon}^x V_\lambda(z) z \Psi_\lambda(z) dz - \Psi_\lambda(x) + \Psi_\lambda(-\epsilon) + (x+\epsilon) \Psi'_\lambda(-\epsilon) = E \int \int \Psi_\lambda dz \quad (4.60)$$

Source (4.61) Spanish PDF page 102; separate display withdrawn.

$$\langle \rho | \Psi \rangle = -\Psi'(+\epsilon) + \Psi'(-\epsilon) = E \int_{-\epsilon}^{+\epsilon} \Psi(y) dy \quad (4.61)$$

Source (4.62) Spanish PDF page 102; separate display withdrawn.

$$\epsilon \langle \rho | \Psi \rangle = -\langle \rho | z \Psi(z) \rangle = -\Psi(\epsilon) + \Psi(-\epsilon) + 2\epsilon \Psi'(-\epsilon) = E \int \int \Psi \quad (4.62)$$

Source (4.65) Spanish PDF page 102; separate display withdrawn.

$$-V = \lambda \delta(x) + \mu \delta'(x) \quad (4.65)$$

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$$-V = \lim_{\epsilon \rightarrow 0} \lambda_+ \delta(x - \epsilon) + \lambda_- \delta(x + \epsilon) + \mu_+ \delta'(x - \epsilon) + \mu_- \delta'(x + \epsilon) \quad (4.66)$$

Source (4.67) Spanish PDF page 103; separate display withdrawn.

$$\Psi'_\epsilon(\epsilon) - \Psi'_\epsilon(-\epsilon) + F(\epsilon) = \lambda_+ \Psi(+\epsilon) + \lambda_- \Psi(-\epsilon) + \mu_+ \Psi'(+\epsilon) + \mu_- \Psi'(-\epsilon) \quad (4.67)$$

Source (4.68) Spanish PDF page 103; separate display withdrawn.

$$\Psi(\epsilon) + \Psi(-\epsilon) + G(\epsilon) = 2\epsilon \lambda_- \Psi(-\epsilon) - 2\epsilon \mu_- \Psi'(-\epsilon) + \mu_+ \Psi(\epsilon) + \mu_- \Psi \quad (4.68)$$

Source (4.69) Spanish PDF page 103; separate display withdrawn.

$$-\Psi'(0^+) = \Psi'(0^-) = \lambda_+ \Psi(0^+) + \lambda_- \Psi(0^-) + \mu_+ \Psi'(0^+) + \mu_- \Psi'(0^-) \quad (4.69)$$

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$$\Psi(0^+) + \Psi(0^-) = +\mu_+ \Psi(0^+) + \mu_- \Psi(0^-) \quad (4.70)$$

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$$|f(\varphi)| \leq C \sum_{\alpha \leq n} \sup_{x \neq 0} |(\frac{d}{dx})^\alpha \varphi|, \varphi \in K, \text{supp}(\varphi) \in X \quad (4.72)$$

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$$\langle 1' | f \rangle = f(+0) - f(-0) \quad (4.73)$$

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$$\langle \psi | \delta \rangle = \langle \delta | \psi \rangle \langle \delta | + \frac{\langle 1' | \psi \rangle}{4} \langle 1' | \quad (4.74)$$

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$$\frac{d}{dx} \Psi(+0) = \frac{X_1}{4} \Psi(+0) \quad \Psi(0^-) = 0 \quad (4.80)$$

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$$\psi(+0) - \frac{d}{dx} \psi(-0) = -X_1/4 \psi(-0) \quad (4.81)$$

Source (4.82) Spanish PDF page 105; separate display withdrawn.

$$-D_x^2 + A\delta + B\delta' \quad (4.82)$$

Source (4.86) Spanish PDF page 106; separate display withdrawn.

$$0 = \langle \rho | \Psi \rangle \quad (4.86)$$

Source (4.87) Spanish PDF page 106; separate display withdrawn.

$$F\psi'(0^\pm) = -\langle \rho | z \Psi(z) \rangle \quad (4.87)$$

Source (4.88) Spanish PDF page 106; separate display withdrawn.

$$H_K = -D_x(1 - F\delta)D_x \quad (4.88)$$

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$$-\langle \delta^2 | 1 \rangle = -\langle \delta | \delta \rangle = -\langle \delta | H' \rangle = \langle D_x \delta | H(x) \rangle = -\langle \delta | (dH/dx) \rangle = -\langle \delta | 0 \rangle = 0 \quad (4.89)$$

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$$\cancel{H_S = -D_x^2 + \lambda|\delta' > \delta'|} \quad (4.90)$$

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$$\cancel{AA^+ = -Z^*(x)\frac{d^2}{dx^2}Z(x) - \frac{d}{dx}Z^*W + W^*Z\frac{d}{dx} + |W|^2} \quad (4.94)$$

Source (4.95) Spanish PDF page 107; separate display withdrawn.

$$\cancel{Z(x) = \begin{cases} x > 0, & e^{i\theta_+} \\ x < 0, & e^{i\theta_-} \end{cases}} \quad (4.95)$$

Source (4.96) Spanish PDF page 107; separate display withdrawn.

$$\cancel{W(x) = \begin{cases} x > 0, & \mu_+ e^{i\theta_+} \\ x < 0, & \mu_- e^{i\theta_-} \end{cases}} \quad (4.96)$$

Source (4.97) Spanish PDF page 108; separate display withdrawn.

$$\cancel{0 = \int ((-Z(x)\phi^*(x))'\Psi(x) - \phi^*(x)Z(x)\Psi'(x))} \quad (4.97)$$

Source (4.98) Spanish PDF page 108; separate display withdrawn.

$$\cancel{= \tilde{Z}(0^+)\phi^*(0^+)\Psi(0^+) - Z(0^+)\phi^*(0^+)\Psi(0^+)} \quad (4.98)$$

Source (4.101) Spanish PDF page 108; separate display withdrawn.

$$\cancel{Z(0^+)\Psi'(0^+) + W(0^+)\Psi(0^+) = \frac{e^{i(\theta_+ - \theta_-)}}{\lambda^*}(Z(0^+)\Psi'(0^+) + W(0^+)\Psi(0^+))} \quad (4.101)$$

Source (4.102) Spanish PDF page 108; separate display withdrawn.

$$\cancel{\Psi'(0^+) + \mu_+\Psi(0^+) = \frac{1}{\lambda^*}\{\Psi'(0^-) + \mu_-\Psi(0^-)\}} \quad (4.102)$$

Source (4.105) Spanish PDF page 108; separate display withdrawn.

$$\cancel{\Psi'(0^+) = \frac{1 + |\lambda|^2}{\lambda^*}\mu_-\Psi(0^-) + \frac{1}{\lambda^*}\Psi'(0^-)} \quad (4.105)$$

Source (4.106) Spanish PDF page 108; separate display withdrawn.

$$\cancel{\Psi(0^+) \equiv \lambda\Psi(0^-)} \quad (4.106)$$

Source (4.107) Spanish PDF page 108; separate display withdrawn.

$$\cancel{\Psi'(0^+) = \frac{1 - |\lambda|^2}{\lambda^*}\mu_-\Psi(0^-) + \frac{1}{\lambda^*}\Psi'(0^-)} \quad (4.107)$$

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$$\Psi(0^\pm) = 0, \quad (4.108)$$

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$$Z(0^-)\Psi'(0^-) + W(0^-)\Psi(0^-) = 0 \quad (4.109)$$

Source (4.110) Spanish PDF page 109; separate display withdrawn.

$$D(A) = AC(\mathbb{R} - \{0\}) \quad (4.110)$$

Source (4.111) Spanish PDF page 109; separate display withdrawn.

$$\Psi'(0^-) + \mu_- \Psi(0^-) = 0 \quad (4.111)$$

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$$\Psi'(0^+) + \mu_+ \Psi(0^+) = 0 \quad (4.112)$$

Source (4.113) Spanish PDF page 109; separate display withdrawn.

$$Z(x) = 1 + \frac{1}{2}\lambda\delta(x) \quad (4.113)$$

Source (4.114) Spanish PDF page 109; separate display withdrawn.

$$A = (1 + \frac{1}{2}\lambda\delta(x)) \frac{d}{dx} \quad (4.114)$$

Source (4.115) Spanish PDF page 109; separate display withdrawn.

$$H_+ = A^+ A = -\frac{d}{dx} (1 + \lambda\delta(x)) \frac{d}{dx} \quad (4.115)$$

Source (4.116) Spanish PDF page 110; separate display withdrawn.

$$\Psi(0^-) = \frac{1 + iA}{1 - iA} \Psi(0^+) \quad (4.116)$$

Source (4.118) Spanish PDF page 110; separate display withdrawn.

$$\langle \Psi | [P, f] | \Psi \rangle = \langle P^+ \Psi | f \Psi \rangle = \langle f^* \Psi | P \Psi \rangle \quad (4.118)$$

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$$= i \left(\int_{-\infty}^0 + \int_0^{\infty} \right) (\Psi'^* f \Psi + \Psi^* f^* \Psi') = \quad (4.119)$$

Source (4.120) Spanish PDF page 110; separate display withdrawn.

$$= i|\Psi|^2 f|_{0+}^{0-} + i \left(\int_{-\infty}^{0-} + \int_{0+}^{\infty} \right) (\Psi^*(f^* - f)\Psi' - |\Psi|f') \quad (4.120)$$

Source (4.121) Spanish PDF page 111; separate display withdrawn.

$$Q = \frac{d}{dx} + \lambda H(x) + \mu \quad (4.121)$$

Source (4.122) Spanish PDF page 111; separate display withdrawn.

$$Q = \lim_{\epsilon \rightarrow 0} \frac{1}{1 + \lambda U_{\epsilon}(x)} \frac{d}{dx} \quad (4.122)$$

Source (4.123) Spanish PDF page 112; separate display withdrawn.

$$[Q, f]\Psi = (Qf)\Psi \quad (4.123)$$

Appendix C

Bibliography concordance

The bibliography retains all 79 entries of the Spanish edition and includes all 49 sources identified in the chapter review. This concordance keeps the original bibliography numbers distinct from the linked reference numbers of this edition. Publication records include the later journal version where the thesis used a preprint.

The original bibliography

Original reference [64] is retained as the attribution of the Salcedo numerical programs. The published numerical method of Oset and Salcedo is listed separately [136].

Spanish	Reference in this edition
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[78]	S. Wolfram [141].
[79]	B. H. Zhao [142].

Sources identified in the chapter review

The R-numbers refer to the review catalogue. The references occur in the relevant scientific discussions in this thesis or in the companion renormalisation article. The location column identifies that discussion; it also keeps the expanded Chapter 5 research in the article.

Review	Reference	Discussion
R1	L. J. Boya, R. F. Wehrhahn and A. Rivero [26]	Thesis.
R2	L. J. Boya and A. Rivero [1]	Thesis and article.
R3	V. A. Marchenko [16]	Thesis.
R4	I. Kay and H. E. Moses [17]	Thesis.
R5	C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura [20]	Thesis.
R6	C. V. Sukumar [18]	Thesis.
R7	P. Deift and X. Zhou [21]	Thesis.
R8	E. P. Wigner [14]	Thesis and article.
R9	J. Kellendonk and S. Richard [8]	Thesis.
R10	L. Infeld and T. E. Hull [28]	Thesis.
R11	M. M. Crum [29]	Thesis.
R12	L. E. Gendenshtein [30]	Thesis.
R13	G. J. Heckman and E. M. Opdam [34]	Thesis.
R14	E. M. Opdam [35]	Thesis.
R15	F. Cooper, A. Khare and U. Sukhatme [31]	Thesis.
R16	P. Lagogiannis, A. Maloney and Y. Wang [37]	Thesis.
R17	M. Isachenkov and V. Schomerus [38]	Thesis.
R18	M. Isachenkov and V. Schomerus [39]	Thesis.
R19	S. Odake and R. Sasaki [36]	Thesis.
R20	M.-A. Miri, M. Heinrich, R. El-Ganainy and D. N. Christodoulides [40]	Thesis.
R21	B. Simon [41]	Thesis and article.
R22	B. Simon [42]	Thesis.
R23	J. Aguilar and J.-M. Combes [47]	Thesis.
R24	E. Hernández, A. Jáuregui and A. Mondragón [48]	Thesis.
R25	R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey and D. E. Knuth [49]	Thesis.
R26	P. Exner and P. Šeba [65]	Thesis.
R27	P. Kurasov [69]	Thesis and article.
R28	T. Cheon and T. Shigehara [62]	Thesis and article.

Review	Reference	Discussion
R29	Yu. D. Golovaty and R. O. Hryniv [60]	Thesis and article.
R30	S. Albeverio, L. Dąbrowski and P. Kurasov [55]	Thesis and article.
R31	T. Fülöp and I. Tsutsui [98]	Thesis.
R32	I. Uchino and I. Tsutsui [70]	Thesis.
R33	M. Asorey, A. Ibort and G. Marmo [56]	Thesis and article.
R34	V. Kostykin and R. Schrader [99]	Thesis.
R35	M. Olshanii [75]	Thesis.
R36	T. Cheon and T. Shigehara [76]	Thesis.
R37	P. Martinetti and R. Wulkenhaar [73]	Thesis.
R38	P. Martinetti [100]	Thesis.
R39	A. Lejay [74]	Thesis.
R40	S. Ohya, M. Sakamoto and M. Tachibana [80]	RG article.
R41	J. L. Cardy [101]	RG article.
R42	I. Affleck and A. W. W. Ludwig [102]	RG article.
R43	C. L. Kane and M. P. A. Fisher [94]	RG article.
R44	V. de Alfaro, S. Fubini and G. Furlan [103]	RG article.
R45	M. C. Birse, J. A. McGovern and K. G. Richardson [81]	RG article.
R46	D. Cafiero, M. Correggi and D. Fermi [96]	RG article.
R47	F. Gesztesy and H. Holden [54]	Thesis.
R48	P. Deift and X. Zhou [97]	Thesis.
R49	P. R. Chernoff and R. J. Hughes [58]	Thesis.

Later connections and editorial provenance

(2026) The following map gathers the later connections in their scientific setting. The marks in the running text distinguish established developments from the additional calculations of this edition.

Chapter 1.

Topological Levinson theory [8], nonlinear inverse scattering and Riemann–Hilbert asymptotics [20, 21], and causality restrictions on range expansions [14, 15].

Chapter 2.

Hypergeometric shift operators [34, 35]; exceptional-polynomial shape invariance [36]; de Sitter transparency [37]; conformal blocks [38, 39]; and supersymmetric optical structures [40].

Chapter 3.

Strong-coupling localisation [118, 42], analytic resonance theory and coalescing poles [47, 105, 48], and Lambert-function branches [49].

Chapter 4.

Resonant zero-range limits [60, 61], general quantum-graph couplings [66, 99], circle contacts and holonomy [98, 100], singular-domain SUSY [70], geometric approximations [67], confined atoms [75], Bose–Fermi duality [76], spectral distance [73] and semipermeable diffusion [74].

Chapter 5 and the companion article.

The thesis chapter retains the original scattering-space argument. The article develops boundary-condition RG [80], short-range effective theory [81], two-dimensional dimensional transmutation [83, 108, 86], singular conformal mechanics [103, 88, 89, 90], many-body boundary flows [101, 102, 104, 94, 95] and the 2026 many-centre limit [96].

(2026-gpt) The additions calculated for this edition include the threshold-state factorisation in Chapter 1, the explicit metric normalisation in Chapter 2, the local double-pole and Lambert-function calculations in Chapter 3, and the finite-link and resistance–distance comparisons in Chapter 4. The article contains the exact matrix-flow synthesis, its diagonal bare-path model, resonant detuning calculations, stable scalar constructions, the gauge completion and the internal-defect and connector actions. These marks record the work done in this edition. The cited historical results retain their own authorship and dates.

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